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ISSUE No.14

PRACTICAL MATHEMATICS

THEORY AND PRACTICE WITH MILITARY AND INDUSTRIAL APPLICATIONS

GENERAL APPLICATIONS

in the fields of

Structural Engineering
Machine-Shop Practice
Heat • Chemistry
Electricity • Gunnery
Map-Reading
Navigation • Aviation
Radio • Vectors

COMPREHENSIVE INDEX

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ISSUE 14 **Practical Mathematics** VOLUME 2

REGINALD STEVENS KIMBALL, *Editor*

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CHATS WITH THE EDITOR

THIS is the last of the issues of PRACTICAL MATHEMATICS devoted to military and industrial applications. When you began this intensive study of mathematical theory and practice about four months ago, some of you were a bit skeptical as to whether or not you could really complete the lessons in the span of 120 days. Now you are in a position to judge for yourself how well you were able to keep to that goal.

As I wrote many of you, and as I intimated in some of my earlier chats, no one is under compulsion to get through the work in any given amount of time. Some of you have been putting the later issues by until you should have had opportunity to give fuller attention to the subject-matter of the earlier issues. Others, for whom the earlier issues were in the nature of a review of work previously studied, more than kept abreast of the dates of issuance through the first section and have then devoted more careful attention to the later issues immediately upon receiving them.

Under the stress of preparing and studying these lessons as rapidly as possible in order to be ready for any emergency growing out of the war effort, we have all had to pick and choose our way. When the officials of the National Educational Alliance approached me early last fall with the proposition that I assist them in organizing this series of lessons and that I have them complete by midsummer, I realized that the complexities of the project would be many.

I wish that it were possible to bring each of you behind the scenes to see just what goes into the making of such a publication as PRACTICAL MATHEMATICS. In addition to our corps of instructors, each working in his own office on his own campus, we have gathered together a headquarters staff to care for the various phases of the publication. Every manuscript received from the professors who wrote them has been checked and re-checked to make sure that nothing was taken for granted in the transition from one writer's presentation to the next. Every problem proposed for solution by the student has been solved independently by at least three members of our staff—and, when their solutions differed, others of the staff have been called into consultation.

Our staff artists have endeavored to prepare diagrams that would really aid in interpreting the text and have sometimes been of assistance in calling attention to obscurities in the text which we have thereupon remedied. In addition, there has been a constant procession of proofs across my desk—first proofs, second proofs, sometimes third proofs, and then lock-ups—with possibilities for further improving the text at each re-appearance of the materials. Our copyholders and proof-readers have not been idle in assuring you a reasonably correct text at all points. The co-operation of the compositors, make-up men, and others on the printer's staff should not go unmentioned.

Before we leave the subject, it seems appropriate that we take a final glance back over the issues, to make certain that we have not overlooked

any of the phases of mathematics on which it is necessary to draw.

In Issue Number One, we devoted most of our attention to basic arithmetic, considering in detail the fundamental operations with whole numbers, fractions, and mixed numbers. We also gave some attention to systems of weights and measures and to the Arabic system of notation, on which most modern mathematical operations are based.

In Issue Number Two, we continued with a consideration of decimals, averages, ratio and proportion. In this issue, also, we were introduced to logarithms and the slide rule, ingenious devices for shortening the labors of mathematical computation.

In Issue Number Three, we met the special problems of algebraic notation, and learned how to perform the fundamental operations with monomials and polynomials, how to make use of factoring as a short form of division, and how to deal with fractions and equations. We learned that many examples which cannot be solved readily by arithmetic may be easily solved when we resort to algebra.

Issue Number Four continued with further algebraic processes, particularly simultaneous equations, quadratics, permutations, combinations, the binomial theorem, number series, roots and powers. We learned also how to make use of various forms of graphs as a short form of solution or as an easier way to make intelligible through an appeal to eye-interest the relationship between numerical data.

In Issue Number Five, we turned to plane geometry, avoiding the formal treatment which is given in most Euclidean texts and proceeding directly to make use of the results of these proofs. A considerable part of the issue was devoted to construction exercises, since this is the phase of geometrical work with which the shop-worker is most concerned. We tied together our knowledge of algebra and geometry in a special article on the conic sections, dealing with the circle, the ellipse, the parabola, and the hyperbola.

In Issue Number Six, we extended our knowledge to the field of solid geometry, learning how to compute the areas and volumes of three-dimensional figures, relating our knowledge to mechanical drawing and construction engineering. We gave further attention to algebra in an article on irrationals and imaginary numbers.

Issue Number Seven was devoted to a presentation of plane trigonometry, with a consideration of right and oblique triangles, trigonometric identities, and inverse functions. We learned that the slide rule is a great help in solving many of the formulas with which we are confronted in dealing with trigonometry.

In Issue Number Eight, we approached the field of the calculus, treating of both differential calculus and integral calculus at sufficient length to learn how we may draw upon this branch of mathematical theory to help us solve problems which involve variables. Two short articles on fractional exponents and complex numbers extended our knowledge of algebra.

In Issue Number Nine, we discussed differential equations, thereby appreciating more fully the help which the calculus can be to us. In this issue, we discussed also the subject of mensuration, at first a seemingly insignificant, but upon consideration a most important, subject. Here we were most concerned with tolerances, significant figures, and precision. Our acquaintance with a great number of instruments which may be used in working out precise measurements was increased through the discussion of their utility in various fields.

Throughout the issues in this first part of our course, we stopped at

intervals to work on puzzle-type problems which gave us a deeper appreciation of the fallacies which sometimes lead us astray when we go wool-gathering in our use of mathematics. The glossaries at the ends of the various issues gave us an opportunity to catch ourselves up if we had missed an important point when it was first introduced.

With the tenth issue, we began to consider the application of mathematics to one or another of the scientific fields in which constant use is made of the theory which had been presented in the earlier issues.

In Issue Number Ten, we considered mathematics as applied to the fields of construction engineering and machine-shop practice. This was followed in Issue Number Eleven with a similar treatment of the fields of heat and chemistry. In Issue Number Twelve, we gave attention to the mathematics of electricity and gunnery. In Issue Number Thirteen, we treated similarly of the fields of navigation and radio. In the present issue, we take a quick review of all of these fields, treating at some length a problem which calls into play our whole range of mathematical knowledge.

If this were a regular course, given on a college campus, we should feel at this point that we were approaching commencement day (graduation day). Since this is not a correspondence course, we have no awarding of final honors, no presentation of diplomas, no salutatory or valedictory speeches. I cannot let this opportunity go by, however, without passing on to you a final word of advice.

I hope that this series of lectures on mathematics has served its purpose of offering you the key by which you may safely pass through the portals of mathematical learning, discovering, as we promised you in the earlier issues, that there is nothing mysterious about mathematical procedures. By this time, you should have come to a full appreciation of the sense of order which is omnipresent in the whole field of mathematics. Half the secret of success lies in setting down in orderly fashion what you already know about the problem on which you are working and then determining how to relate these known facts so that you may be able to determine other facts from them. The remainder of the secret lies in correct application of the formulas, particularly in being sure that you have made your computations accurately, whether you have worked them out "long hand" or whether you have relied upon tables or other devices to shorten your labor.

There is no royal road to learning. No one has ever yet devised a completely painless process of passing on the lore of the ages to a new generation. Diligence and hard labor are ever the lot of the conscientious student, in no matter what field he may be working. Particularly is this true in the field of mathematics. We may be perplexed at times, almost baffled at times, yet a rigid adherence to the principle of constant daily practice helps us to go steadily upward toward our goal.

It is our hope that the arrangement of these fourteen issues may have been such that we have been able to assist our Alliance members in their progress through the field of mathematics. My colleagues and I have sought in our presentation of the various fields to make the approaches as gradual, the passage from one stage to another as little abrupt as possible, within the limits of space allotted to us. If you can approach the next problem on which you have to work with a little greater confidence, with more assurance that you are going to come out with the cor-

rect solution, then our labors have not been in vain.

In 120 days, we have covered the range of mathematical theory insofar as it relates to an elementary consideration of those fields which have a direct bearing on present-day military and industrial problems. We have not endeavored to cover the whole field of mathematical knowledge. That will have to be left for some future occasion when time is less pressing, a time which we hope is, for most of you, in the not too distant future. If we have whetted your interest in mathematics and put you in a position where mathematics becomes your servant and not your master, then, we can assure you, you have before you many pleasant years of further study.

To those of you who are ending your association with us at this point, let us bid you a pleasant farewell. To those who are planning to continue with us in the further pursuance of mathematical interests, we part for a time, to renew our associations in another interesting series of lectures in the near future.

R.S.K.

ABOUT OUR AUTHORS

DAVID R. CURTISS was born in Derby, Connecticut, in 1878. He obtained his Bachelor's and Master's degrees at the University of California, and in 1903 received his Doctor's degree from Harvard. He spent the next year as an "interne" at the École Normale Supérieure in Paris, and then became an instructor in mathematics in the Sheffield Scientific School of Yale University. In 1905 he joined the faculty of Northwestern University, and has been a Professor of Mathematics there since 1909. At various times, he has served as visiting professor at the University of Chicago, Harvard University, and the University of California.

Dr. Curtiss is a member of a number of learned societies, having served as Vice President of the American Mathematical Society and of the American Association for the Advancement of Science, and as President of the Mathematical Association of America. He has contributed many articles to mathematical periodicals and has written a number of mathematical texts, including several on trigonometry.

Since 1912, Dr. Curtiss has served continuously as an editor of mathematical journals and texts, including six years as an editor of the *Transactions* of the American Mathematical Society, and ten years as an editor of the *Bulletin* of that Society.

· GENERAL APPLICATIONS ·

By D. R. Curtiss, Ph.D.

AS a final review for the course in PRACTICAL MATHEMATICS with military and industrial applications, we shall present in this article a project from each of the applied fields covered in previous sections. In studying the theory, most students find that they can, by dint of more or less diligent labor, follow the line of reasoning of an author. In the more general treatment of practical fields in the preceding four sections, the reader was enabled to draw together, from the theory which preceded, the mathematical formulas necessary for the solution of the basic formulas presented.

In this concluding section, we shall review each of the applied fields, taking from each a specific problem which will call into play a considerable part of the whole range of mathematical knowledge. In following through the various steps, we shall assume that the reader is somewhat familiar with the subjects involved, although explanation of the derivation of certain formulas is included for purposes of making it possible for those less familiar to follow the line of reasoning.

APPLICATIONS TO STRUCTURAL ENGINEERING

As we learned in the article on construction engineering, for every stress in a member there is a corresponding strain. We learned how to compute the bending moment of a beam, thus being enabled to select the smallest beam commensurate with safety. We shall now extend our investigation to the subject of deflection.

While a beam may be perfectly safe, it will still bend, even though imperceptibly, under any load. This sag is shown in Fig. 1 in a greatly exaggerated manner, where the distance, y, represents the deflection.

The allowable deflection in engineering practice is $\frac{1}{360}$ of the span.

A greater amount leads to a feeling of insecurity and results in an excessive cracking of floor and ceiling surfaces.

While the amount of deflection for a given set of circumstances may be computed from formulas given in standard handbooks, that man who understands their derivation will be able to avoid the grotesque errors which sometimes arise from their use.

To develop the formula for deflection in a simple beam, uniformly loaded, we return to the formulas,

$$\frac{M}{f}=\frac{I}{c} \quad \text{or} \quad \frac{M_x}{I}=\frac{f}{c},$$

each of which is equal to a third ratio, $\frac{E}{R}$.

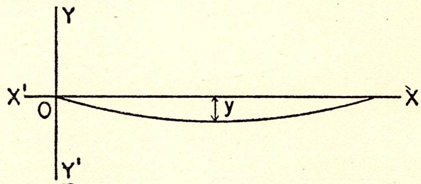


Fig. 1

a From the form,

$$\frac{M_x}{I}=\frac{E}{R},$$

we derive our basic formula,

$$M_x=\frac{EI}{R},$$

where M_x = bending moment at any distance, x, from left reaction
I = moment of inertia
E = modulus of elasticity
R = radius of curvature.

Curvature (R) may be defined for any curve as $\frac{\Delta\theta}{\Delta s}$ where $\Delta\theta$ is the change of angular direction in radians over the distance of arc, Δs.

The result will be exact if we let Δs approach 0, in which case we should have $\frac{d\theta}{ds}$.

b From the treatment on page 485, it should now be evident that, if we let Δx approach 0, Δx becomes dx, Δy becomes dy, and Δs becomes ds.

By geometry,

$$ds^2=dx^2+dy^2.$$

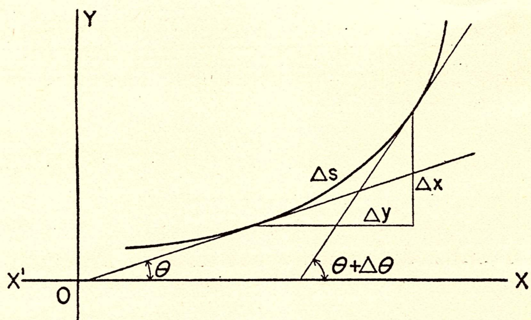


Fig. 2

c Dividing by dx, taking the square root of each side, and then solving for ds, we have:

$$ds=\sqrt{1\left(\frac{dy}{dx}\right)^2}\,dx$$

d Since the tangent to a curve at any point, X, is given by the first derivative, $\frac{dy}{dx}$,

$$\tan\theta=\frac{dy}{dx}$$

$$\theta=\tan^{-1}\frac{dy}{dx}$$

$$d\theta=\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

e Substituting the values of ds and $d\theta$ found in steps c and d, we get

$$K=\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

f By trigonometry, we know that an arc length divided by the angle in radians is equal to the radius of a circle for that arc and angle. The radius of curvature, R, is therefore defined as $\frac{1}{K}$,

$$R=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

g Since deflection in a beam is very small, the angle will be close to 0 and so will be tangent, or the first derivative. The square of this quantity will be altogether negligible. Therefore, in this case, the equation reduces to

$$R=\frac{1}{\frac{d^2y}{dx^2}}\quad\text{or}\quad\frac{1}{R}=\frac{d^2y}{dx^2}.$$

h Substituting this value for $\frac{1}{R}$ in the equation of step a,

$$M_x=EI\frac{d^2y}{dx^2},$$

which is known as the *differential equation of the elastic curve.*

i Fig. 3 shows in Cartesian coördinates a beam with uniform loading of w lb. per ft. The total load will be lw and each reaction will be $\frac{l}{2}w$. At $\frac{l}{2}$, $\frac{dy}{dx}=0$.

The maximum deflection occurs at this point; hence, it is usually for this value of y that we solve the equation. Let us, however, first solve the more general equation for a bending moment at x:

$$M_x=\frac{-wx^2}{2}+\frac{lwx}{2}.$$

j Substituting this value in the equation of step h, and remembering that E and I are constants, we have:

$$EI\frac{d^2y}{dx^2}=\frac{-wx^2}{2}+\frac{lwx}{2}$$

$$EI\frac{dy}{dx}=\frac{-wx^3}{6}+\frac{lwx^2}{4}+C_1$$

k When $\frac{dy}{dx}=0$, $x=\frac{l}{2}$.

$$C_1=\frac{l^3w}{48}-\frac{l^3w}{16}=\frac{-l^3w}{24}.$$

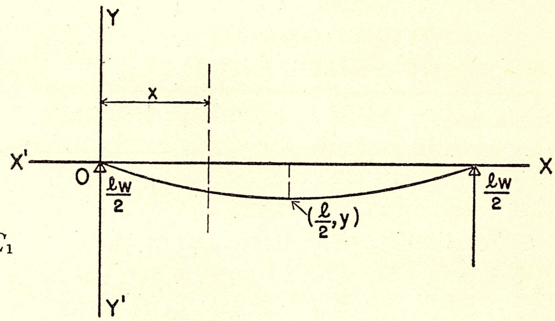


Fig. 3

l Substituting this value in the equation of step j, we find:

$$EIy=\frac{-wx^3}{6}+\frac{lwx^2}{4}-\frac{l^3w}{24}.$$

A second integration gives us

$$EIy=\frac{-wx^4}{24}+\frac{wlx^3}{12}-\frac{-wl^3x}{24}+C_2.$$

m When $x=0$, $y=0$; hence, $C_2=0$.

n The final equation may now be written:

$$Y=\frac{-w}{24EI}(x^4-2lx^3+l^3x).$$

Illustrative Problem

Assume that you have selected a beam for a span of 60′ and that you intend to provide the proper lateral support. The total uniform load will be 1,250 lb. per ft. So far as the bending moment is concerned, the beam is perfectly safe. Its moment of inertia, I, is 6,354.7 in.^{4} E, of course, is 30×10^6.

a The maximum allowable value will be $\frac{1}{360}$ of the span, L,

or $$\frac{60\times12}{360}=2''.$$

b The maximum deflection will occur at the center of the beam; hence,

$$X=30\times12=360''.$$

(Notice that, since I is expressed in inches, all linear measurements must be in the same unit.)

c Since, with this loading, y is a maximum at $x=l$, and the total load, W,

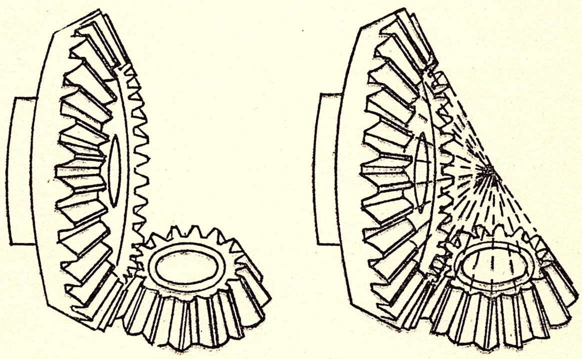
is equal to wl, we may simplify the equation in step n of the preceding theory to

$$y=\frac{-5wl^3}{384EI}=\frac{5\times 75{,}000\times 60^3\times 12^3}{384\times 30\times 10^6\times 6{,}354.7}=1.91''.$$

As this is less than the allowable maximum of 2.0″, the beam is satisfactory.

APPLICATIONS TO MACHINE-SHOP PRACTICE

The article on machine-shop practice (pages 598 to 635) presented many formulas to use in specific situations. While the average workman is probably content to learn the specific formula for the specific purpose, the man with aspirations to get ahead in the shop will find it desirable to work out for himself the design of a new part. For purposes of illustration, we are working out in detail the steps in designing a bevel gear, such as that shown in Fig. 4.



(a) (b)

Fig. 4

Nomenclature

In getting clearly in mind the names of the various parts of the gear which we shall have to use in our problem, we shall find it simpler to begin with an ordinary spur gear. In Fig. 5,

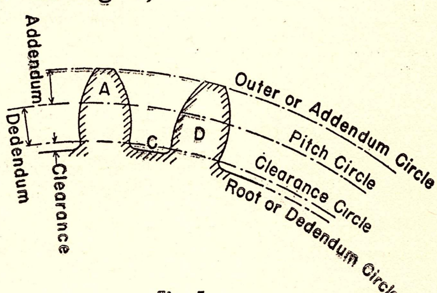


Fig. 5

A = the addendum, the distance from the top of the tooth to the pitch circle, measured along the diameter
C = the clearance, the amount by which the dedendum must exceed the addendum
D = the dedendum, the distance from the root, or dedendum circle, along the diameter
P.C. = the pitch circle, the circumference of the disk
P.D. = the pitch diameter, the diameter of the pitch circle
D.P. = the diametral pitch, the number of teeth per inch of pitch diameter (or the number of teeth within π inches of the pitch circumference
N = number of teeth on the gear

From these, we can establish the relationships,

$$\text{pitch diameter}=\frac{\text{number of teeth}}{\text{diametral pitch}}\qquad \text{P.D.}=\frac{N}{\text{D.P.}}$$

$$\text{addendum} = \frac{1}{\text{diametral pitch}} \qquad A = \frac{1}{\text{D.P.}} \qquad \textbf{b}$$

$$\text{dedendum} = \frac{1.157}{\text{diametral pitch}} \qquad D = \frac{1.157}{\text{D.P.}} \qquad \textbf{c}$$

To determine the outside diameter of the gear (the diameter of the addendum circle), we add twice the length of the addendum to the pitch diameter.

$$\text{O.D.} = \text{P.D.} + 2\text{A}$$

When it comes to a bevel gear, we measure the addendum and the dedendum, as shown in Fig. 6, along the edge of the tooth. For purposes of computation, we may regard a pair of bevel gears as the frustums (see page 344) of two cones with a common vertex and with the lateral surfaces making a line contact along the pitch-cone radius (Fig. 4b).

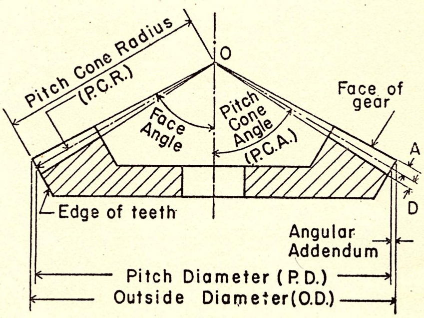


Fig. 6

Illustrative Problem

Suppose that, for a gear with P.D. = 6 and D.P. = 10, you are asked to find the outside diameter, the pitch-cone angle, the pitch-cone radius, the addendum angle, and the dedendum angle. At the same time, you must find the pitch diameter, the diametral pitch, and the parts named above for another gear at right angles to the original gear and with double its angular velocity.

a The angular velocities (ω) of the gear and the pinion will be inversely proportional to their pitch diameters (to the number of teeth).

$$N_g = (\text{P.D.}_g)(\text{D.P.}_g) = 6 \times 10 = 60 \text{ teeth}$$

$$\frac{\omega_g}{\omega_p} = \frac{1}{2}$$

$$\frac{N_g}{N_p} = \frac{2}{1}$$

Then $N_p = 30$ teeth

$\text{P.D.}_p = 3$ inches

$\text{D.P.}_p = \text{D.P.}_g = 10.$

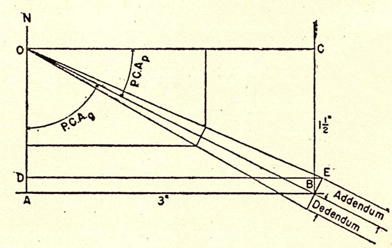


Fig. 7

b Turning to Fig. 7, erect on the vertical center line, NA, a perpendicular, AB, with a length of 3 inches. This line, AB, represents the pitch circle radius of the gear.

c At B, erect BC perpendicular to AB, with a length of $1\frac{1}{2}''$. BC is the pitch circle radius of the pinion (the driven gear).

d Draw OB to represent the pitch-cone radius.

$$\text{P.C.A.}_g = \tan^{-1} 2 \qquad \text{P.C.A.}_p = \tan^{-1}\frac{1}{2}$$

It follows from trigonometry (pages 427, 428) that

$$\text{P.C.A.}_g = 63°26' \qquad \text{P.C.A.}_p = 26°34'$$

Then, by inspection,

$$\text{P.C.R.} = 3 \sec 26°34' = 1.5 \sec 63°26' = 3.35''$$

e Returning to our earlier formulas, and submitting values now known, we have

$$A = \frac{1}{\text{D.P.}} = \frac{1}{10} = 0.1000''$$

$$D = \frac{1.157}{\text{D.P.}} = \frac{1.157}{10} = 0.1157''$$

f Now we are ready to find the addendum angle and the dedendum angle.

$$\angle \text{ add} = \tan^{-1}\frac{0.1000}{3.35}$$

By logarithms, (pages 90, 91)

$$\begin{aligned} \log 0.1000 &= 9.00000-10 \\ -\log 3.35 &= -0.52505 \\ \log \angle \text{ add} &= 8.47495-10 \end{aligned}$$

Referring to a table of logarithms of trigonometric functions (pages 444ff.),

$$\log^{-1} \angle\text{add} = 1°43'$$

Similarly,

$$\angle \text{ ded} = \tan^{-1}\frac{0.1157}{3.35}$$

$$\begin{aligned} \log 0.1157 &= 9.06333-10 \\ -\log 3.35 &= -0.52505 \\ \log \angle \text{ ded} &= 8.53828-10 \end{aligned}$$

$$\log^{-1} 8.53828-10 = 1°59'.$$

These values will be the same for both gear and pinion.

g To find the outside diameter, we must first find the angular addendum, since

$$\text{O.D.} = \text{P.D.} + 2 \text{ A.A.}$$

In Fig. 8, the sides of $\angle\theta$ are perpendicular to $\angle\text{P.C.A.}_g$.

$$\angle\theta = \angle\text{P.C.A.}_g$$

and

$$\text{A.A.}_g = A \cos \text{P.C.A.}_g$$

$$= \frac{1}{10}\cos 63°30' = 0.045''$$

$$\text{O.D.}_g = 6\ (2\times 0.045) = 6.09''$$

$$\text{A.A.}_p = A \sin \text{P.C.A.}_g$$

$$= \frac{1}{10}\sin 63°30' = 0.087''$$

$$\text{O.D.}_p = 3\ (2\times 0.087) = 3.17''.$$

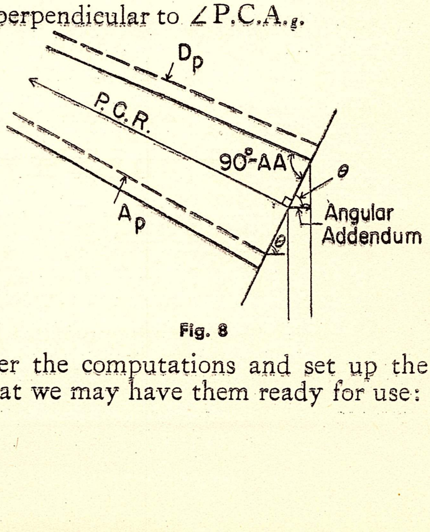


Fig. 8

h As a final step, let us go back over the computations and set up the specifications in tabular form, so that we may have them ready for use:

Specifications

	GEAR	PINION
Outside diameter	6.09″	3.17″
Pitch diameter	6.00″	3.00″
Diametral pitch	10	10
Pitch-cone angle	63°30′	26°30′
Pitch-cone radius	3.35″	3.35″
Addendum angle	1°43′	1°43′
Dedendum angle	1°59′	1°59′

APPLICATIONS TO HEAT

Most of us are familiar with the use of thermostats to set in operation a heating unit to supply heat for raising the temperature and, when the desired temperature has been reached, to break contact, thereby opening the relay and shutting off the heat. Calculating the construction of a thermostat calls into play a number of interesting mathematical formulas.

Illustrative Problem

Construct a thermostat which will touch off a relay at any temperature between 60° and 80° F. to which it may be set. (The instrument should be sufficiently sensitive to respond to a change of 0.5 degree—that is, it should actuate the relay within 0.5° of the temperature for which it is set.) The voltage available for operating the relay is 40 volts.

a The temperature-responsive unit is to be a bimetallic strip of invar-nichrome 0.04″ thick. The overall dimensions of the unit are not to exceed 1.5″ in width nor 4″ in length. (Fig. 9.)

If at temperature t_0 the strip, a solid welded piece, is straight, the strip will "belly" at a higher temperature, t_1, because the nichrome will be slightly longer than the invar. At a certain temperature, the strip will just make contact at *C*. This contact can be made to close a circuit to the relay.

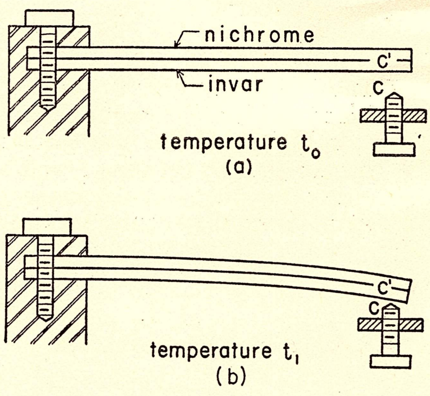


Fig. 9

b The voltage which will pass through the circuit to be opened and closed at *CC′* is 40 v. Since we wish to be sure of positive opening and closing of the circuit for a difference of 0.5°, we must set our tolerance much closer.

Using 5 as a factor of safety, we set the temperature tolerance at 0.1°.

c The thickness of air which will resist arcing of 40 v is about $\frac{1}{1850}''$.

Hence, a temperature change of 0.1° must produce a difference of at least 0.00054″ in the distance, CC', or 0.0054″ per degree.

d We obtain an estimate of the curvature of the strip thus:

In Fig. 10, we represent a segment of the strip which is exactly 1 unit long at the temperature, t_0, for which the strip is straight.

We take the thickness of the invar portion and of the nichrome portion separately.

Fig. 11 shows schematically the curvature of this same segment at a higher temperature, t_1.

The dotted lines drawn through the middle of each semi-strip are called *neutral axes*.

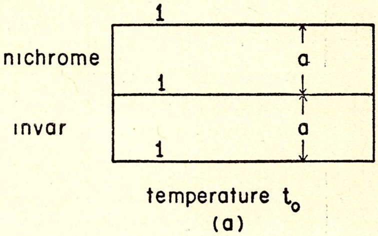


Fig. 10

While the inner arc of the invar semi-strip is shorter than its outer arc, the average is its length along the neutral axis, S_i.

The average length of the nichrome semi-strip is S_n.

At the temperature, t_2, S_n will be greater than S_i because the coefficient of expansion, e_n, of the nichrome is greater than the coefficient of expansion, e_i, of the invar.

$$S_n = 1 + e_n(t_1 - t_0)$$
$$S_i = 1 + e_i(t_2 - t_0)$$

These are the lengths which the two segments would assume as a result of the expansion due to heat. In actual fact, there will be elastic resistance to the expansion, but we shall neglect that for the moment.

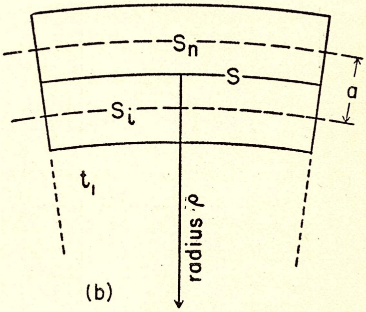


Fig. 11

e Thus, we can derive an expression which will give the radius of curvature, ρ, in Fig. 11. Define ρ as the radius to the arc midway between S_i and S_n, and put a for the semi-thickness of the strip. Since the angular measures of the arcs, S_i and S_n, are the same,

$$\frac{S_i}{\rho - \frac{1}{2}a} = \frac{S_n}{\rho + \frac{1}{2}a}$$

f Solving for $\frac{1}{\rho}$, the curvature,

$$\rho\,(S_n - S_i) = \frac{1}{2}a\,(S_n + S_i)$$

$$\frac{1}{\rho} = \frac{2(S_n - S_i)}{a(S_n + S_i)}$$

which, upon substituting the values in step d, becomes

$$\frac{1}{\rho} = \frac{2(e_n - e_i)\,(t_1 - t_0)}{a(2 + [e_n + e_i]\,[t_1 - t_0]\,)}$$

g Since the factor, $\frac{2}{2 + [e_n + e_i]\,[t_1 - t_0]}$, can never differ from 1, within

the range of our problem, by as much as 1 part in 1000, our solution will be almost as accurate if we write

$$\frac{1}{\rho}=\frac{(e_n-e_i)\,(t_1-t_0)}{a}.$$

h As the coefficient, $\frac{e_n-e_i}{a}$, is constant for the strip, the curvature is proportional to the difference in temperarure, t_1-t_0. We designate this coefficient as b, whence

$$\frac{1}{\rho}=b\,(t_1-t_0).$$

i In this case,
$$e_i=0.000001$$
$$e_n=0.000020$$
$$a=0.02$$
$$b=\frac{0.000020-0.000001}{0.02}=0.00095$$

Then $$\frac{1}{\rho}=0.00095\,(t_1-t_0).$$

j If the strip is fixed at one end, as in Fig. 9, the displacement at the other end may be calculated from the curvature. Here we resort to simple geometry.

In Fig. 12,

OC = the original, or undeflected, position of the strip

l = length of the strip

OC' = an arc of curvature $\frac{1}{\rho}$, representing the deflected strip.

The distance, h, between C and C' is very nearly perpendicular to OC. So also is the line, CR, the extended radius through C' (The inclination of the line is purposely exaggerated in the diagram.)

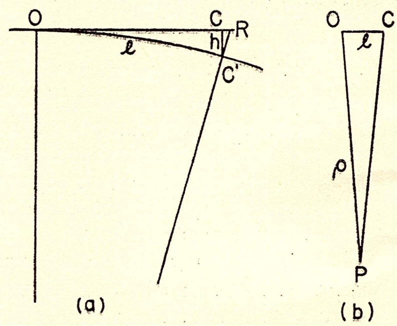


Fig. 12

The triangle, OCP, where P is the center of curvature and $OP=\rho$, is shown increased in size in Fig. 12b, where

$$\rho^2+l^2=CP^2.$$

From Fig. 12a, $$C'R=CP-\rho.$$

Then, since we have agreed to consider $CC'=CR$,

$$CC'=CP-\rho=(\rho^2+l^2)^{+\frac{1}{2}}-\rho=\rho\left(1+\frac{l^2}{\rho^2}\right)^{\frac{1}{2}}-1=h.$$

Expanding the power of the binomial, we get

$$h=\rho\left[1+\frac{1}{2}\,\frac{l^2}{\rho^2}+\frac{1}{2}\left(\frac{1}{2}-1\right)\frac{1}{2!}\,\frac{l}{\rho^4}+\,\ldots\,-1\right]=\frac{l^2}{2\rho}\left[1-\frac{1}{4}\,\frac{l^2}{\rho^2}+\,\ldots\,\right]$$

k The powers of $\frac{l}{\rho}$ inside the brackets in this equation compose a series

which converges to a sum very small compared with 1. Hence, we have an accurate approximation when we write:

$$h=\frac{l^2}{2\rho}.$$

l Using the value of $\frac{1}{\rho}$ in step h, this becomes

$$h=\frac{1}{2}bl^2(t_1-t_0).$$

Since b and l are constant, the motion of the free end of the strip is thus found to be approximately in proportion to the temperature change.

A strip of nichrome-invar 4″ in length and 0.04″ thick would yield for the coefficient of t_1-t_0

$\frac{1}{2}\cdot 0.00095 \cdot 16=0.0076''$ per degree C. or about 0.0042″ per degree F.

Recalling the requirements we are aiming to fill, we find that such a strip would be insufficient. (We require a motion of 0.0054″ per degree of temperature increase plus an allowance of one-fourth or one-third for the neglected effect of elasticity.) Finding, however, that the order of magnitudes is nearly what we require, we cast about for a method of modifying the operation of the device to secure a slightly greater response. An arrangement in the form shown in Fig. 13 suggests itself.

m The bimetallic strip, S, is in the form of a circular arc. One end is riveted at F to a fixed standard. At the other end, a finger, K, made of some conducting metal, is riveted at R, perpendicular to the arc. Temperature changes affecting the radius of the circle produce changes in the position of R, causing motion of the finger, K, which amplifies the response.

Putting ρ_1 for the radius of the circle at temperature t_1, we may obtain, by the same derivation as before

$$\frac{1}{\rho_1}=b(t_1-t_0)$$

where now, however, t_0 is a parameter, not necessarily a temperature at which we can arrive.

Putting $\frac{1}{\rho_2}$ for the curvature of temperature t_2,

$$\frac{1}{\rho_2}=b(t_2-t_0)$$

By subtraction,

$$\frac{1}{\rho_2}-\frac{1}{\rho_1}=b(t_2-t_1).$$

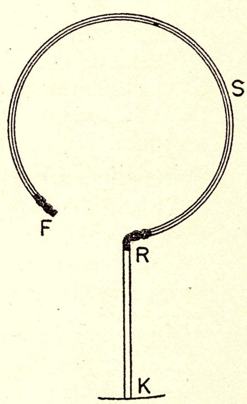


Fig. 13

n The geometry of the arrangement in Fig. 13 is shown in Fig. 14, where $KF \perp AF$ at F.

Lay off radii O_1F and $O_2F=\rho_1$ and ρ_2.

With these radii, draw circles tangent to AF at F, which represents the fixed end of the strip.

The free end of the strip at the temperature, t_1, is at the point, R_1, located on the larger circle at the point which is at a distance of l from F along the circular arc which passes the long way around. (Here l is the length of the strip.)

At the temperature, t_2, the free end of the strip will be at R_2, the point on the smaller circle an arc distance l from F, measured the long way.

Then the finger will reach to K_1, on radius O_1R_1 extended, at temperature t_1, and to K_2, on radius O_2R_2 extended, at temperature t_2.

$K_1R_1 = K_2R_2$, since both equal the length, k, of the finger.

o Designate the angle swept over by k in passing from K_1 to K_2 (the angle between O_1R_1 and O_2R_2, $\angle R_1GR_2$, where G is the point of intersection of the radii extended the other way) as θ.

Writing $\widehat{FR_1}$ for the short arc between F and R, we have:

$$\widehat{FR_1} + l = 2\pi\rho_1$$

$$\widehat{FR_2} + l = 2\pi\rho_2$$

Hence,

$$\angle FO_1R_1 = \frac{\widehat{FR_1}}{\rho_1} = 2\pi - \frac{l}{\rho_1}$$

$$\angle FO_2R_2 = \frac{\widehat{FR_2}}{\rho_2} = 2\pi - \frac{l}{\rho_2}$$

and

$$\angle FO_1R_1 - \angle FO_2R_2 = l\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) = bl\,(t_2 - t_1) = \theta.$$

p The length, r, of the line-segment between G and K is composed of the three segments, k, ρ, and GO_1. Since GO_1 is relatively short, approaching O as FO_1R_1 approaches 0, we may neglect it and write

$$r = \rho + k,$$

dropping the subscripts from r and ρ where their differentials are not involved.

The arc which the length, l, of the bimetallic strip forms about the circle with radius of ρ is a little shorter than the circumference, $2\pi\rho$, the amount of deficiency depending upon the size of $\angle FO_2R_2$. Since this angle is to be as near 0 as construction will permit, we may write the approximation,

$$l = 6\rho,$$

6ρ being just a little shorter than $2\pi\rho$.

The path traversed by the end of the finger in passing from K_1 to K_2 is virtually an arc, θ, of a circle about G with a radius of r. Putting d for the length of the arc traversed,

$$d = r\theta$$

which, from step o, $\quad = brl(t_2 - t_1)$

and from p and q $\quad = 6b\rho(\rho + k)\ (t_2 - t_1)$.

This equation is similar to the one developed in step l, but gives the motion of the arrangement of Fig. 13 for the same temperature change. In order to see what difference exists in the response per degree of temperature due to the use of a circular arc instead of a straight strip, we should put the over-all dimension of Fig. 13 (stripping it to the circular arc by putting $k=0$) for l in the earlier equation.

q Since the over-all dimension is 2ρ, putting $l = 2\rho$ in the equation of step l, we have $2\rho^2$ for the coefficient of $t_2 - t_1$, whereas the coefficient in the present step is $6\rho^2$.

Thus, the response per degree of the circular arc is 3 times (ideally π times) the response of the straight strip.

r If the finger, k, is also made of the bimetallic material, the response factor due to its heating, $\frac{1}{2}k^2$, is added; thus, for this case, the displacement, d_3, is given by

$$d_3=6b\rho(\rho+k)\,(t_2-t_1)+\frac{1}{2}\,bk^2(t_2-t_1)=6b\left(\rho^2+\rho k+\frac{1}{12}\,k^2\right)(t_2-t_1).$$

s Let us see how this form of element will work out in the given problem. We must choose ρ and k within the condition that 2ρ may not exceed 1.5″, the over-all width, and $2\rho+k$ must not exceed 4″, the over-all length. It is easy to see that, under these conditions, the largest value of the factor, $\rho^2+\rho k+\frac{1}{12}k^2$, occurs for $\rho=1$.

Writing down the condition,

$$2\rho+k=4$$

and differentiating with regard to ρ,

$$2+\frac{dk}{d\rho}=0$$

$$\frac{dk}{d\rho}=-2.$$

The derivative of $\rho^2+\rho k+\frac{1}{12}k^2$ is

$$2\rho+k+\rho\,\frac{dk}{d\rho}+\frac{1}{6}\,k\,\frac{dk}{d\rho}$$

or, using $\frac{dk}{d\rho}=-2$,

$$+\frac{2}{3}\,k.$$

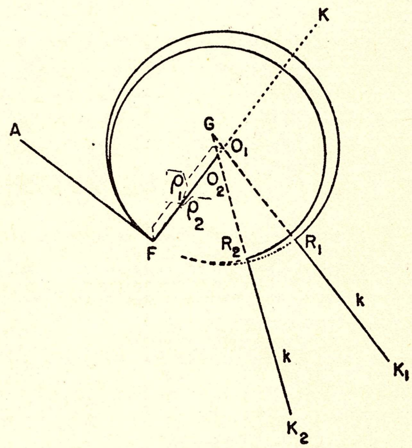


Fig. 14

Since the derivative with respect to ρ is always positive, the greatest value of the factor occurs at the largest permissible value of ρ, which is 0.75″. The corresponding largest permissible value of k is 2.5″.

t We therefore obtain a circle with radius of 0.75″ and a finger 2.5″ long as our temperature responsive element.

Thus, $\rho^2+\rho k+\frac{1}{12}\,k^2=2.57.$

Since $b=0.00095$, the equation in step r reduces to $d_3=0.0074\,(t_2-t_1)$. The motion is 0.0147″ per degree Centigrade or 0.0081″ per degree Fahrenheit. Allowing a reduction in this motion of $\frac{1}{4}$ for the effect of the elasticity, we shall still have left 0.0060″ per degree Fahrenheit. Thus, we have now met the condition imposed for a temperature tolerance with a motion of not less than 0.0054″ per degree Fahrenheit.

u The element appears as in Fig. 15, where the screw, S, has a large head graduated for temperature (and insulated from the screw proper). It is convenient if less than one full rotation of the head covers the entire temperature range. Since the specific temperature range is 60° to 80° F., the span is 20° F.

At 0.0060″ per degree Fahrenheit, this results in a total range of distance SK, equal to

$$20\times0.0060=0.12''.$$

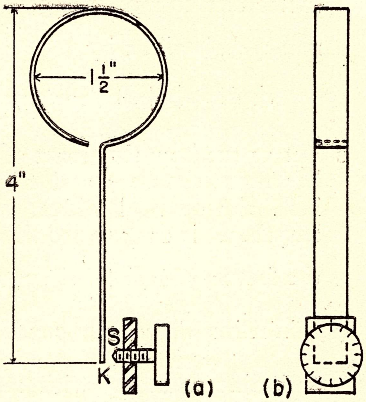


Fig. 15

A screw with 6 threads to the inch will therefore be satisfactory for S. One complete revolution of the screw will give a motion of 0.167″ to S, and the required temperature range of 20° will therefore be covered by 0.72 of one revolution. Then 1° F. will be represented by 0.036 revolution, or 13° on the head of the screw. (By attaching the screw head to the screw proper by means of a friction set-screw, we can calibrate the temperature scale on the screw head.) In actual graduation of the screw head, temperature tests are made to obtain the exact scale. We have here computed the scale with sufficient accuracy to ensure that the parts made to the calculated dimensions will operate together within the desired limits.

APPLICATIONS TO HEAT AND CHEMISTRY

An interesting problem which combines the principles studied in the articles on heat and chemistry is presented in the following example. While the problem consists of a number of steps, it is solved almost entirely by the use of simple arithmetic.

Illustrative Problem

Power for a 200-Kw generator is to be furnished by a steam turbine. The over-all efficiency of the plant will be 20%. The cross-sectional area of the chimney to be erected for the disposal of flue gases will be determined by their volume. Compute this volume, given the following data:

Flue gas temperature	600°	
Excess air supply	50%	
Coal analysis:		
High-grade bituminous		
$\frac{\text{B.T.U.}}{\text{lb.}}$	14,000	
Moisture	2%	by weight
Ash	5%	by weight
Carbon	77%	by weight
Sulphur	1%	by weight
Hydrogen	5%	by weight
Nitrogen	1%	by weight
Oxygen	9%	by weight
	100%	

a Since the efficiency of the plant is given as 20%, we proceed on the assumption that we must design for 5 times the quantity desired.

$$200 \text{ Kw} \times 5 = 1{,}000 \text{ Kw}.$$

b Knowing that there are 746 watts in 1 H.P. we convert Kw to H.P. by multiplying by $\frac{1{,}000}{746}$: $\frac{1{,}000 \times 1{,}000}{746} = 1{,}320$ H.P.

c Making use of the fact that 1 H.P. will furnish 33,000 ft.-lb. per minute and that 778 ft.-lb. equal 1 B.T.U., we can now compute the number of B.T.U. required per minute.

$$\frac{1{,}320 \times 33{,}000}{778} = 56{,}000 \ \frac{\text{B.T.U.}}{\text{min.}}$$

d We find from the data that 1 lb. of coal provides 14,000 B.T.U. Therefore, the coal consumption will be:

$$\frac{56{,}000}{14{,}000} = 4 \ \frac{\text{lb.}}{\text{min.}}$$

e Using this figure with the original data, we proceed to figure the weight of each constituent of the coal:

Moisture	$0.02 \times 4 = 0.08$ lb.
Ash	$0.05 \times 4 = 0.20$ lb.
Carbon	$0.77 \times 4 = 3.08$ lb.
Sulphur	$0.01 \times 4 = 0.04$ lb.
Hydrogen	$0.05 \times 4 = 0.20$ lb.
Nitrogen	$0.01 \times 4 = 0.04$ lb.
Oxygen	$0.09 \times 4 = 0.36$ lb.
TOTAL	4.00 lb.

f Since the ash produces no heat, we shall be no longer concerned with that item. The oxygen and the nitrogen, also, have no heating value, but we shall need to consider them further in computing the volume of the flue gases. To that end, we first find the weights of the products of combustion, obtaining these balanced equations:

$$\begin{array}{ccc} 3.08 \text{ lb.} & & \text{lb.} \\ C & + O_2 \rightarrow & CO_2 \\ 12 & & 44 \end{array}$$

$$\begin{array}{ccc} 0.04 \text{ lb.} & & \text{lb.} \\ S & + O_2 \rightarrow & SO_2 \\ 32 & & 64 \end{array}$$

$$\begin{array}{ccc} 0.2 \text{ lb.} & & \text{lb.} \\ 2H_2 & + O_2 \rightarrow & 2H_2O \\ 4 & & 36 \end{array}$$

g From these, we get: $CO_2 = \frac{44}{12} \times 3.08 = 11.29$ lb.

$$SO_2 = \frac{64}{32} \times 0.04 = 0.08 \text{ lb.}$$

$$H_2O = \frac{36}{4} \times 0.20 = 1.80 \text{ lb.}$$

$$13.17 \text{ lb.}$$

h Where did this additional weight come from? The weight of the coal, as found in step e, was only 4 lb. The difference between the weight of the three gases formed and the weight of the three elements from which they were formed is made up of the oxygen, which combined with those elements to form oxides.

To provide for complete combustion of the fuel, an excess of air is required. Too little air results in the formation of carbon monoxide, CO, instead of carbon dioxide, CO_2. Since CO produces only one-third the amount of heat produced by CO_2, we naturally want to provide sufficient air to get the best results.

The excess of air, however, should be kept within the smallest possible bounds because the unused oxygen, and all the inactive nitrogen, must be heated, with resultant waste in thermal energy.

From step e, the weight of our three elements is:

$$\underset{3.08}{\text{Carbon}} + \underset{0.04}{\text{Sulphur}} + \underset{0.20}{\text{Hydrogen}} = 3.32 \text{ lb.}$$

Subtracting this from the weight of the oxides, as found in step g, gives us

$$13.17-3.32=9.85 \text{ lb.}$$

i Since the excess of air must provide 50% more oxygen than is theoretically necessary, we must have

$$9.85\ \frac{9.85}{2} = 9.85+4.93=14.78 \text{ lb.},$$

the total oxygen required. (The 4.93 lb. of oxygen remains uncombined in the flue gas.)

j By reference again to step e, we discover that the amount of oxygen supplied by the coal is only 0.36 lb. This means that

$$14.78-0.36=14.42 \text{ lb.}$$

is to be provided by the air supply.

k Since oxygen represents only 23% of air by weight, the total air supply must be:

$$14.42\times\frac{100}{23}=62.7 \text{ lb.}$$

l The difference between this weight and the weight of the oxygen (step j) will represent nitrogen. (Other gases, such as argon, are present, but in such small quantities that we can neglect them in this type of computation.)

$$62.7-14.42=48.28 \text{ lb.}$$

m Adding the weight of the nitrogen present in the fuel itself (step e), we get

$$48.28+0.04=48.32 \text{ lb.}$$

n From the burning of the fuel, 1.80 lb. of water (H_2O) results. To this should be added the 0.08 lb. of moisture present in the fuel (step e).

$$1.80+0.08=1.88 \text{ lb.}$$

o Before proceeding to find the volumes of the gases, let us tabulate the results of the previous steps:

Flue Gases

CO_2	11.29 lb.	Step g
SO_2	0.08 lb.	
H_2O	1.88 lb.	Step n
O_2	4.93 lb.	Step i
N_2	48.32 lb.	Step m

p These weights must now be converted into volumes at 600° F. and standard pressure. We learned on page 674 that a gram-molecule of any gas occupies 22.4 liters at 0° C. and standard pressure.

Transformed to the English system of measurement, this would mean that a lb.-molecule of any gas at 32° F. and standard pressure would occupy 358.7 cu. ft.

We remember also (page 671), that, with pressure constant, the volume of a given quantity of gas is proportional to its absolute temperature. In the Fahrenheit scale, we obtain absolute temperature by adding 460° to the standard temperature reading.

$$T_g = 600° + 460° = 1060°$$
$$T_0 = 32° + 460° = 492°$$

We may now set up our equations for volumes:

$$\text{Volume } CO_2 = \frac{11.29}{44} \times 358.7 \times \frac{1060}{492} = 197 \text{ cu. ft.}$$

$$\text{Volume } SO_2 = \frac{0.08}{64} \times 358.7 \times \frac{1060}{492} = 1 \text{ cu. ft.}$$

$$\text{Volume } H_2O = \frac{1.88}{18} \times 358.7 \times \frac{1060}{492} = 81 \text{ cu. ft.}$$

$$\text{Volume } O_2 = \frac{4.93}{32} \times 358.7 \times \frac{1060}{492} = 119 \text{ cu. ft.}$$

$$\text{Volume } N_2 = \frac{48.32}{28} \times 358.7 \times \frac{1060}{492} = \underline{1{,}330} \text{ cu. ft.}$$

TOTAL 1,728 cu. ft. per min.

(The slide rule or logarithms are a great help here!)

q An approximate value for the diameter of the chimney may be obtained from the formula:

$$D = \frac{1}{10}\sqrt{\frac{\text{Volume of flue gas per hour}}{65}}$$

Volume per hour $= 1728 \times 60 =$ 103,680

$$D = \frac{1}{10}\sqrt{\frac{103{,}680}{65}} = 3.95 \text{ or approximately } 4'.$$

APPLICATIONS TO ELECTRICITY Transmission of power over long distances is effected by means of alternating current—not by direct current. This is because of the ease with which the voltage of A.-C. current may be transferred from the thousands of volts in the transmission lines to the 110 volts used in supplying houses with current.

Illustrative Problem

A plant erected 30 miles from an electric generating station will require a power line connecting it to the station. The total horse power of the equipment installed in the plant is 1250, all electrically driven. The efficiency of the electric transmission system in the plant is reckoned at 75% of the total power supplied to the plant. The losses in transmitting the electric power from the generating station to the plant are to be held to 10%. Determine the voltage

at which the power is to be transmitted, 50 tons of copper wire having been allotted to the construction of the line.

a Let Q = the power requirements of the plant.

Then H.P. = $Q \times$ transmission efficiency within the plant.

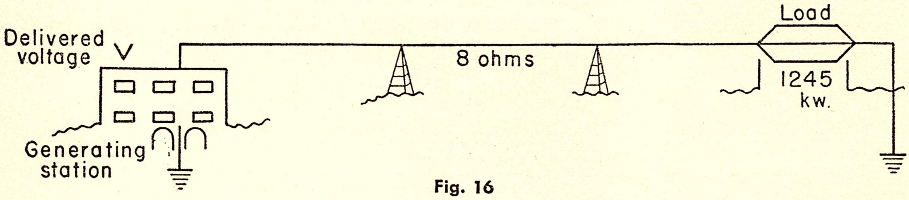


Fig. 16

If Q is to be expressed in kilowatts, the installed H.P. should be converted into kilowatts before being equated.

$$1 \text{ H.P.} = 746 \text{ w} = 0.746 \text{ Kw.}$$
$$0.75\,Q = 1250 \times 0.746 \text{ Kw.}$$
$$Q = 1245 \text{ Kw,}$$

the amount of power to be delivered in the plant, after line losses in transmission but before the losses suffered in the plant.

b The power cannot be transmitted at the "load" voltage (ranging for ordinary equipment from 100 to 1000 volts) because of the following facts:

(*i*) The 50 tons (100,000 pounds) of copper, distributed over 30 miles (158,400 feet), will give a strand weighing 631 pounds per 1000 feet.

By reference to Table LXXI (page 762), we find that this approximates gauge 0000 and that the resistance is about 0.05 ohm per 1000 feet.

The total resistance of the 158,400 feet of line in such a case will, therefore, be about

$$\frac{158{,}400}{1{,}000} \times 0.05 = 7.92 \text{ (approximately 8) ohms.}$$

(*ii*) If the total plant load, Q (= 1245 Kw), were to be delivered at 1000 volts, this would obviously call for a current of 1245 amperes, four or five times the safe carrying capacity of the wire with which we are dealing. Furthermore, the voltage drop over the line resistance of 8 ohms for a current of 1245 amperes would be nearly 100,000 volts.

Actually, only about 1 ampere would pass through the circuit under the given conditions, and the line losses (the energy transformed into heat as the voltage dropped along the transmission line) would absorb over 90% of the energy delivered to the line.

(*iii*) Through a transformer at the generating station, we must step up the voltage of the electric power which is to be delivered to such a level that the line loss will amount to only 10% of the power delivered.

c As the first step in determining the voltage, in Fig. 16, let

V = voltage at the power station
I = current through the circuit
E = voltage drop along the transmission line (between the power station and the load)
P = total power delivered to the circuit.

Then, since $I = \dfrac{E}{R}$, $I = \dfrac{E}{8}$ amperes;

Multiplying I by E, we get the line loss,

$$EI=\frac{E^2}{8}\text{ watts}=1.25\,E^2\cdot 10^{-4}\text{ Kw.}$$

d The load is 1245 Kw. Hence, the total power delivered to the circuit is

$$P=1245+1.25\,E^2\cdot 10^{-4}\text{ Kw.}$$

e By specification, the line loss is to be 10% of the total power delivered. Thus,

$$\begin{aligned}1.25\,E^2\cdot 10^{-4}&=\frac{1245+1.25\,E^2\cdot 10^{-4}}{10}\\ 12.5\,E^2\cdot 10^{-4}&=1245+1.25\,E^2\cdot 10^{-4}\\ (12.5\,E^2-1.25\,E^2)10^{-4}&=1245\\ 11.25\,E^2\cdot 10^{-4}&=1245\\ 10^{-4}\cdot E^2&=\frac{1245}{11.25}=110.7\\ E^2&=1.05\text{ Kv}=1{,}050\text{ v,}\end{aligned}$$

the voltage drop in the line transmission.

f Substituting this value of E in the formula developed in step c, we get

$$I=\frac{1050}{8}=131\,a$$

g We may express the total load of the circuit in terms of the station voltage and current by

$$P=VI\text{ w.}$$

h Substituting for E in the formula of step d, we have

$$P=1245+138=1383\text{ Kw.}$$

i The term, 138, is (to the extent of the three significant figures retained) equal to the 10% of the total load. This checks the equation, the term being the line loss.

j Substituting for P and I in the equation of step g, we get

$$\begin{aligned}1383&=131\,V\\ V&=10.5\text{ Kv.}\end{aligned}$$

A standard 1100-volt transformer will be required. This permits keeping the power losses in the transmission line down to 10% of the total power delivered to the line.

APPLICATIONS TO GUNNERY

One of the difficult factors in ballistics is to analyze the part played by wind friction. Actual computations in the field cannot afford to ignore this important factor. In the following problem, we shall, however, ignore it for the time being.

Illustrative Problem

A 75-mm. gun, firing HE shell, is located with respect to the observer, O, at 1,100 yd., back-azimuth 270. Using the nomenclature of Fig. 27, page 746, we have the following data:

OT=4,899 yd., azimuth 000

OP=3,000 yd., azimuth 120.

We are to find the firing angle, A, and the distance from the gun to the target, GT, by graphic and by analytic methods.

a Using a scale of 1 inch to 1,000 yards, we lay off TM (Fig. 17), and choose on it some convenient place for point O.

Since OT has an azimuth of 000, or 0°, it will lie on TM. T is located at a distance of 4.899″ from O on TM. The back azimuth of OG is given as 270. The azimuth is equal to this value less 180° or 90°. This means that OG makes a 90° angle with OT, measured clockwise. Therefore, we lay off OG perpendicular to TM, with a length of 1.1″ in accordance with our scale.

b To locate the line, OP, we use a protractor, finding an angle of 120° to correspond with the given azimuth.

On this line, at a distance of 3″ from O, we mark the point, P, to represent the given distance of 3,000 yd.

Then we connect PG and TG.

By the use of the protractor, we find the angle between these two lines to be 145°.

Since this is the angle, A, the first part of our problem has been solved.

c We now measure GT and find it to be 5″, representing a distance of 5,000 yd. Our problem has now been solved graphically.

d Analytically, we proceed by drawing a sketch (Fig. 18) like that in Fig. 17, but not necessarily to scale. On it, we place all available information.

e The distance, GT, is easily obtainable by the Pythagorean theorem (page 268):

$$GT = \sqrt{1{,}000^2 + 4{,}899^2}$$
$$= 5{,}000 \text{ yd.}$$

f The firing angle,

$$\angle TGP = 360° - \angle TGO - \angle PGO.$$

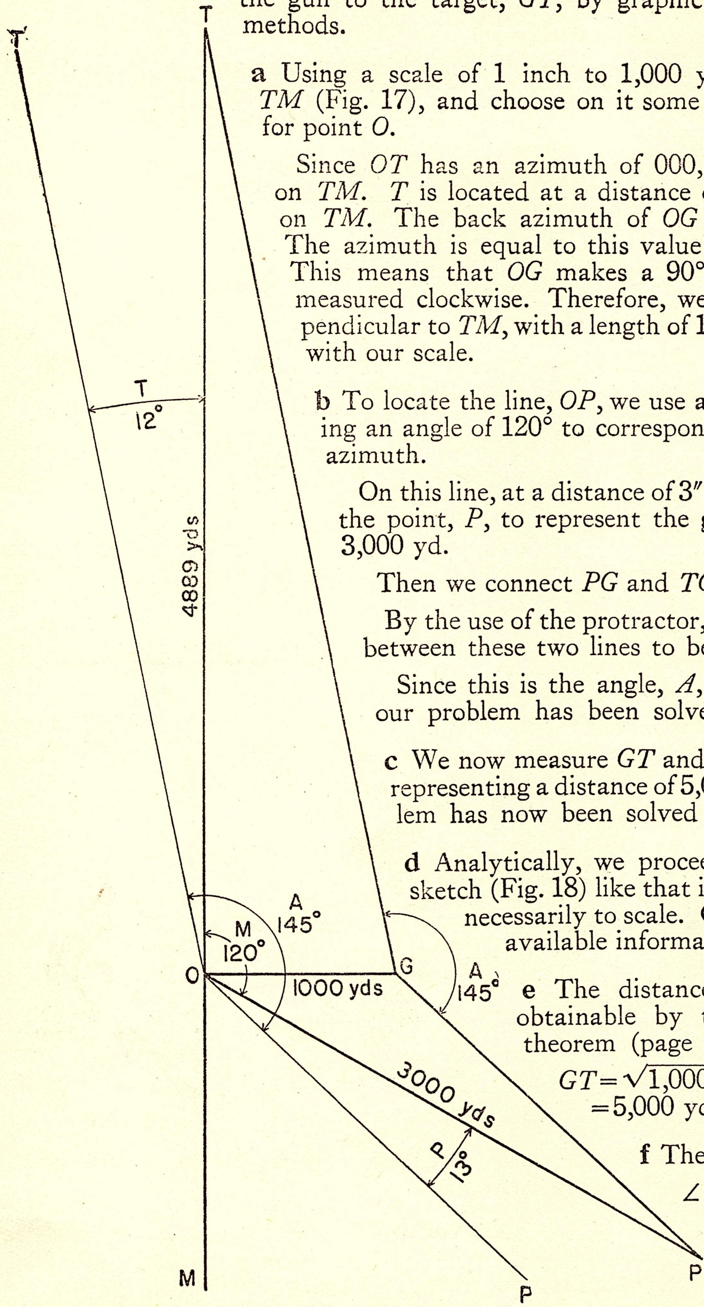


Fig. 17

g We obtain the value of $\angle TGO$ from the trigonometric relationship,

$$\cos \angle TGO = \frac{1{,}000}{5{,}000} = 0.20000.$$

$$TGO = \cos^{-1} 0.20000 = 78°5'.$$

h To find $\angle OGP$ most easily, we resort to the law of tangents (page 417):

$$\frac{\tan \frac{1}{2}(\beta-\gamma)}{\tan \frac{1}{2}(\beta+\gamma)} = \frac{b-c}{b+c} = \frac{3{,}000-1{,}000}{3{,}000+1{,}000} = \frac{2{,}000}{4{,}000} = 0.50000.$$

$$\beta+\gamma = 180° - 30° = 150°.$$

$$\frac{1}{2}(\beta+\gamma) = 75°.$$

$$\tan 75° = 3.73205.$$

By substitution,

$$\tan \frac{1}{2}(\beta-\gamma) = \frac{2{,}000}{4{,}000} \times 3.73205 = 1.86603.$$

$$\frac{1}{2}(\beta-\gamma) = 61°45'.$$

$$\beta-\gamma = 123°30'.$$

We are interested in $\angle\beta$:

$$\begin{array}{r} \beta-\gamma = 123°30' \\ \underline{\beta+\gamma = 150°} \\ 2\beta = 275°30' \end{array}$$

$$\beta = 136°45' = \angle OGP.$$

i $\angle TGP = 360° - 78.5° - 136°45' = 144.75°$ or $144°45'$.

This agrees very closely with the graphic result; hence, we may assume our analysis to be correct.

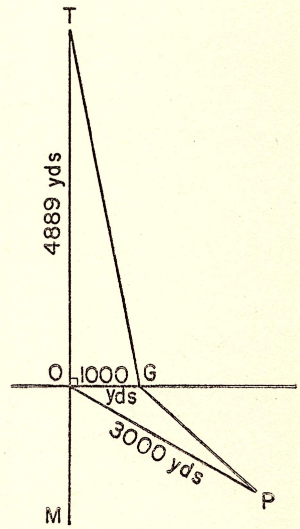


Fig. 18

j By means of the formulas on page 750,

$$R = \frac{V^2 \sin 2\phi}{g} \qquad 15{,}000 = \frac{(1755)^2 \sin 2\phi}{32.2} \qquad \sin 2\phi = \frac{32.2 \times 15{,}000}{(1755)^2}.$$

By logarithms,

$$\begin{array}{ll} \log 32.2 & = 1.50786 \\ \log 15{,}000 & = 4.17609 \\ \text{colog } (1755)^2 & = \underline{3.51145-10} \\ \log \sin 2\phi & = 9.19540-10 \end{array}$$

$$\begin{array}{rl} \log 1755 = & 3.24427 \\ \log (1755)^2 = 2 \log 1755 = & 6.48855 \\ & 10.00000-10 \\ & \underline{-\ 6.48855} \\ \text{colog } (1755)^2 = & 3.51145-10 \end{array}$$

From the table of logarithms of trigonometric functions, we find that the corresponding angle has a value of 9°.

Then $$\phi = \frac{9°}{2} = 4.5°.$$

The time of flight is found from the formula,

$$T = \frac{2v \sin \phi}{g} = \frac{2 \times 1755 \times \sin 4.5°}{32.2}.$$

Resorting once more to logarithms,

$$\begin{aligned} \log 3510 &= 3.54531 \\ \log \sin 4.5° &= 8.89464-10 \\ \text{colog } 32.2 &= \underline{8.49214-10} \\ & \quad 20.93209-20 \\ \log T &= 0.93209 \\ T &= 8.55'' \end{aligned}$$

k We now wish to find whether or not the projectile will clear a hill whose crest is high above a point 3000 yd. from G on GT.

To find the equation of y, we make use of equation III on page 750:

$$\begin{aligned} y &= x \tan \phi - \frac{16.1}{V_0{}^2 \cos \phi} x^2 \\ &= 9000 \tan 4.5° - \frac{16.1 \times 81{,}000{,}000}{(1755)^2 \cos^2 4.5°} \\ &= 706 - \text{fraction as above} \end{aligned}$$

By logs,

$$\begin{aligned} \log 16.1 &= 1.20683 \\ \log 81{,}000{,}000 &= 7.90849 \\ \text{colog } (1755)^2 &= 3.51145-10 \\ \text{colog } \cos 4.5° &= \underline{0.00268} \\ & \quad 12.62945-10 \\ \log \text{fraction} &= 2.62945 \\ \text{fraction} &= 426. \end{aligned}$$

Then $y = 706 - 426 = 280'$.

Since the hill is 500′ high, we shall, obviously, have to find a different trajectory.

l As shown on page 751, we use the complementary angle, $90° - 4.5° = 85.5°$.

Then
$$\begin{aligned} y &= 9{,}000 \times 12.706205 - \frac{16.1 \times 81{,}000{,}000}{(1755)^2 \cos^2 85.5°} \\ &= 114{,}356 - \text{fraction as above} \end{aligned}$$

By logs,

$$\begin{aligned} \log 16.1 &= 1.20683 \\ \log 18{,}000{,}000 &= 7.90849 \\ \text{colog } (1755)^2 &= 3.51145-10 \\ \text{colog } 85.5° &= \underline{2.21070} \\ & \quad 14.84784-10 \\ \log \text{fraction} &= 4.84784 \end{aligned}$$

Then $y = 114{,}356 - 70{,}540 = 43{,}816'$.

m This will obviously clear the target. It is interesting to point out that the time of flight is greatly increased.

$$T = \frac{2v \sin \phi}{g} = \frac{2 \times 1755 \times \sin 85.5°}{32.2}$$

$$\begin{aligned} \log 3510 &= 3.54531 \\ \log \sin 85.5° &= 9.99866-10 \\ \text{colog } 32.2 &= \underline{8.49214-10} \\ & \quad 22.03611-20 \\ \log T &= 2.03611 \\ T = 109'' &= 1.81' \end{aligned}$$

As pointed out in the article on gunnery (page 752), the actual range of the gun for a given elevation might, because of air resistance, be only

70% of the theoretical. Reference to a firing table shows that, with the data, and with an elevation of 4.5°, the actual range would be only 3,000 yd., just 60% of the theoretical value found above.

APPLICATIONS TO MAP-READING Most of us have been familiar with maps from the days when we first studied geography, but many people find difficulty, when confronted with a map, in visualizing the area which the map represents. The clue to our understanding of the map lies in our ability to interpret what the map stands for.

In the corner of almost every map, we find a "scale of miles", which is placed there to help us make this interpretation. (Fig. 19.) In comparing one map with another, we need to note whether or not the two maps are drawn to the same scale.

Opening at random an atlas which stands on a table at my elbow, I note a map of the United States drawn to a scale of 1 : 12,500,000, a map of South America drawn to a scale of 1 : 23,500,000, and a map of Australia drawn to a scale of 1 : 26,000,000. Simply looking from one map to another will not give me a clear concept of the relative sizes of the geographic units portrayed. I must translate these scales into something comprehensible.

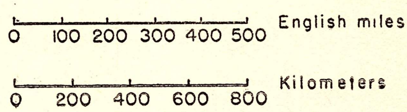


Fig. 19

Scale drawing

We can best get at this matter by first attempting to make a scalar representation of an area which we can readily see. For instance, the library reading-room in the PRACTICAL MATHEMATICS office is approximately 15′ by 25′. There are two doors, one on the north side and one on the west side, and four large windows, all on the south side. If we wanted to send you a life-sized floor plan, we should have to use a sheet of paper far larger than could conveniently be folded up to insert in the magazine. By resorting to a scale drawing, we can readily represent to you the proportions between the various parts of the room named. By letting $\frac{1''}{16} = 1'$, we could reduce the drawing to the size shown in Fig. 20. This is a very small drawing, however. By taking $\frac{1''}{4} = 1'$, we can still give you a drawing small enough to fit readily into the compass of this article and yet large enough to be read with some degree of accuracy, as in Fig. 21. Comparison of Figs. 20 and 21

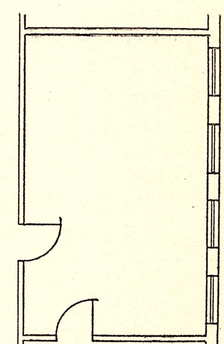
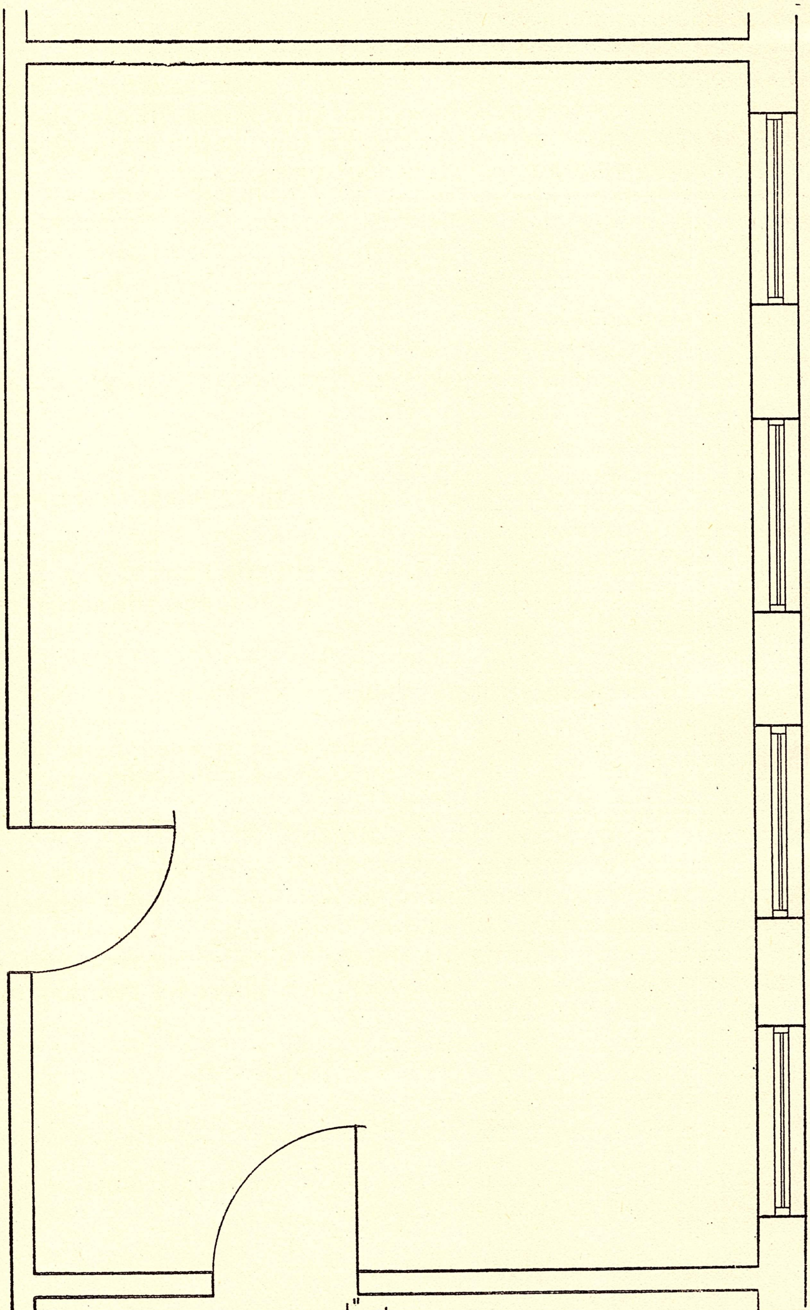

Scale : $\frac{1''}{16} = 1'$

Fig. 20



Scale: $\frac{1}{4}'' = 1'$

Fig. 21

shows a similarity of shape and of location of the various parts. Since Fig. 21 is done at a scale which is four times as large as Fig. 20, one might suppose that Fig. 21 would be four times as large as Fig. 20. Inspection of the two figures shows that such is not the case, however. Fig. 21 is sixteen times as large as Fig. 20, since both the length and the width of the drawing are increased four times.

$$A_2 = l_2 \times w_2 \qquad \text{a}$$
$$A_3 = l_3 \times w_3 \qquad \text{b}$$
$$l_3 = l_2 \times 4 \qquad \text{c}$$
$$w_3 = w_2 \times 4 \qquad \text{d}$$

Substituting c and d in b,

$$A_3(l_2 \times 4)\ (w_2 \times 4)$$
$$4l_2 \times 4w_2$$
$$16(l_2 \times w_2)$$
$$16A_2.$$

Since Fig. 20 is drawn at a scale of $\frac{1''}{16} = 1'$, it is evident that the ratio, reducing feet to sixteenths of an inch, is 1:192. The actual room, then, is $(192)^2$ as large as the drawing. In the case of Fig. 21, the scale is 1:48, reducing feet to fourths of an inch, and the actual room is $(48)^2$ as large as the drawing.

Transition to map-reading

Obviously, in the case of maps, the scale is even more greatly reduced. When we use 1 inch to represent 1 mile (which is 63,360 inches), the map is drawn to a scale $\frac{1}{63360}$ as great and so represents an area which is $(63360)^2$ the size of the paper on which it is drawn, or we may state it thus: 1 square inch on the map represents 4,014,489,600 square inches of land or water area.

The fractional form of reporting the scale is called the representative fraction (abbreviated R.F.). Other common scales for military maps are

6 inches to 1 mile $\frac{1}{10560}$

25 inches to 1 mile $\frac{10}{25344}$

The most common scale, used in France, Germany, most of continental Europe and the United States is

1 cm. to 1 Km. $\frac{1}{100{,}000}$.

Given the scale of the map, you can readily work out for yourself its R.F.; given the R.F., by reversing the process, you can just as readily work out the scale of miles.

Use of the military protractor

In reading or marking directions on a map, we are greatly assisted

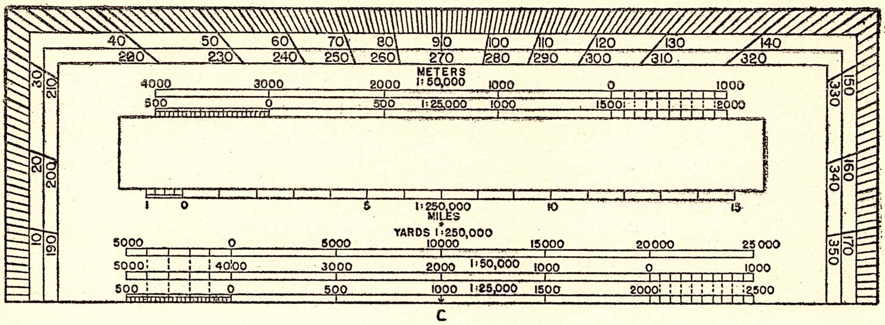


Fig. 22

by the use of the so-called military protractor (Fig. 22).

The graduations along three edges are accompanied by two sets of figures, one reading from 0° to 180° and the other from 180° to 360°.

In aviation and navigation work, directions are always read from the North, clockwise. Hence, to read any angle from 0° to 180°, we place the protractor with its non-graduated edge on the right-hand side of the North-South line, while, to read angles greater than 180°, we place the protractor with its non-graduated edge on the left-hand side of the North-South line.

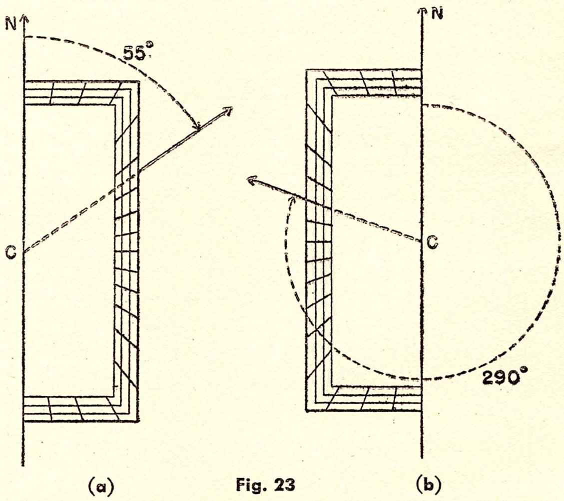


Fig. 23

Having located the desired spot, we connect it with the center of the non-graduated side (indicated in Fig. 22 as *C*), and, with the aid of a ruler or straight edge, note the point at which the straight line crosses a graduation-mark on the protractor.

Thus, to locate 55°, we should proceed as in Fig. 23a; to locate 290°, we should proceed as in Fig. 23b.

These protractors usually have additional scales making it possible to measure on the map exact distances. These scales resemble those already mentioned in connection with Fig. 19, and the method of

using them is similar. Care should be taken to use that scale which is drawn to the same R.F. as the map used.

APPLICATIONS TO NAVIGATION

The broad subject of navigation fills many ponderous tomes. In the previous article on the subject in PRACTICAL MATHEMATICS, we considered some of the mathematical bearings of the subject. In the present problem, we shall see some of the interesting ways in which mathematics is brought into play in the solution of an actual problem.

Illustrative Problem

A ship sails from St. John, New Brunswick (Lat. 45°15′ N, Long. 66°09′ W) on September 4, 1943, bound for New York. Its course is to pass east of Nantucket (Lat. 41°15′ N, Long. 69°57′ W). The time of departure, 11 P.M. $= 23^h$. The speed for the first hour (in the harbor) is $7\frac{1}{2}$ knots; thereafter, 12 knots. The weather reports indicate fog in the sound. As a result, the course is set to clear Nantucket by 16 miles.

a First we determine the bearing of the course (middle latitude).

Difference in longitude (St. John to Nantucket)	3°48′ = 228′	
Middle latitude	43°15′	
Cosine middle latitude		0.72837
Departure (228×0.728)	166 mi.	
	− 16	
	150	
Difference in latitude	240	
tan (180° − bearing)		0.62500
Bearing (180°+32°)	212°	

Distance to point 16 mi. E of Nantucket, 240 sec 32° = 283 mi.

Approximate distance to be sailed in first 20 hours = 240 mi.

b In determining the position on the first day at sea, the navigator first determines times and suitable points for observation:

i About one-half hour before sunrise

ii The noon sun

iii In the afternoon

iv About one-half hour after sunset.

The ship's watch is set 4^h30^m fast of Greenwich Civil Time, corresponding with Local Civil Time at 67°30′ W. The ship's chronometer is on G.C.T. (by radio).

The following data from the *Nautical Almanac* show bodies visible at the desired times:

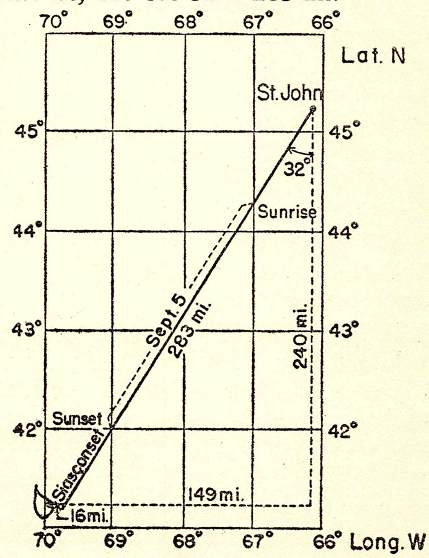


Fig. 24

TABLE A

Approximate longitude, 68° *W*

Object	Local Civil Time (*Watch*)	Greenwich Civil Time (*Chronometer*)	Declination	Greenwich Hour Angle (*approx.*)	Local Hour Angle (*approx.*)
Polaris	5^h	9^h30^m	+89°	100°	32° W
Jupiter	5^h	9^h30^m	+17°	349°	79° E
Sunrise	5^h30^m	10^h	+ 7°	330°	98° E
Moonrise	10^h54^m	15^h24^m	−13°	347°	81° E
Noon sun	12^h	16^h30^m	+ 7°	68°	0°
Sun	14^h30^m	19^h	+ 7°	105°	37° W
Moon	14^h30^m	19^h	−13°	39°	29° E
Sunset	18^h29^m	22^h59^m	+ 7°	158°	90° W
Polaris	19^h	23^h30^m	+89°	310°	118° E
Alpheratz	19^h	23^h30^m	+29°	336°	92° E

Jupiter, with a somewhat greater hour angle and greater declination than the sun, will rise sooner. Thus, Jupiter and Polaris may be used to give a position at 5 A.M. The sun and the moon will both be visible in the early afternoon. Just after sunset, Polaris and Alpheratz will be visible, with Alpheratz toward the east.

i The first observations are taken at 5^h05^m on the watch (chronometer 9^h35^m G.C.T.) and give the altitude for Polaris (corrected for instrument error, but not for refraction and horizon dip) 45°13′.9 and of Jupiter 21°08′.8.

TABLE B

OBSERVATIONS

September 5, 1943. G.C.T.—Watch time = 4^h30^m

Object	Watch	Observer's Altitude	Quadrant
Polaris	5^h05^m	45°13′.9	N
Jupiter	5^h05^m	21°08′.8	E
Noon sun (L.L.)*	12^h01^m	53°37′.9	S
Sun (L.L.)	14^h30^m	41°24′.8	SW
Moon (L.L.)	14^h35^m	27°44′.7	SE
Polaris	19^h00^m	41°38′.0	N
Alpheratz	19^h00^m	16°43′.7	NE

* L.L. = lower limb

Before reducing these observations to lines of position, we wish to know our dead reckoning position at the time of the observations, 5^h05^m. This is computed in the first column of Table C and found to be 67°00′ W, 44°17′ N.

We determine our latitude from the Polaris observation by correcting the observed latitude for refraction and horizon dip (Table D, column 1) and then applying the correction depending on the hour angle (Polaris being not exactly at the pole), taken from the *Nautical Almanac*. The latitude determined from the observation is 44°18′.5 N. (The latitude by dead reckoning had been 44°17′.0 N.)

To plot the Sumner line for the Jupiter observation, we need the altitude and azimuth of Jupiter computed for some assumed near-by position.

POSITION PLOTTING SHEET

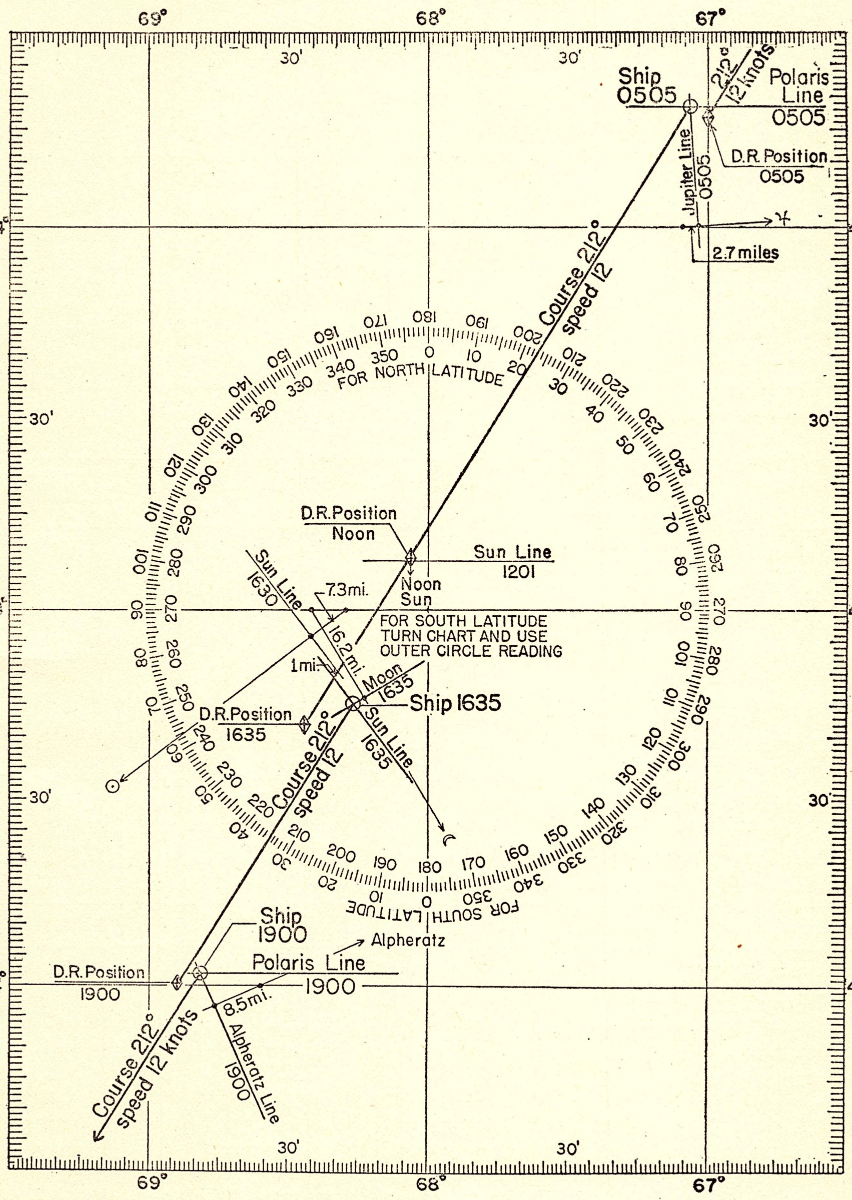


Fig. 25

TABLE C

COMPUTATION OF DEAD RECKONING POSITIONS

September 5, 1943.

Speed : 12 knots. Bearing of course 212° = C cos C = 0.848
sin C = 0.530

Run	1	Noon	2	3
Initial time	....	0505	0505	1435
Initial longitude W	66°09′	67°03′.4	67°03′.4	68°15′.9
Initial latitude N	45°15′	44°18′.5	42°18′.5	42°45′.2
Elapsed time	....	$6\frac{11}{12}$ hr.	$9\frac{1}{2}$ hr.	$4\frac{5}{12}$ hr.
d = Distance	68 .5	83	114	53
d cos C = Δ Lat.	−58′.0	−70′.4	−96′.6	−44′.9
d sin C = Departure	36.3 mi.	44.0 mi.	60.4 mi.	28.1 mi.
Middle latitute	44°46′.0	43°43′.3	43°30′.2	42°22′.8
cos Mid. Lat.	0.7100	0.7227	0.7253	0.7387
Δ Longitude*	51′.1	61′.0	83′.3	38′.1
D.R. longitude	67°00′.1	68°04′.4	68°26′.7	68°54′.0
D.R. latitude	44°17′.0	43°08′.1	42°41′.9	42°00′.3
Final time	0505	1200	1435	1900

*Δ Longitude = departure ÷ cos middle latitude

TABLE D

SIGHTS FOR LATITUDE

Object Observed	Polaris (*Dawn*)	Noon Sun	Polaris (*Evening*)
Watch time	5^h05^m	12^h01^m2	19^h00^m
G.C.T.—W.T.	4^h30^m	4^h30^m	4^h30^m
Greenwich Civil Time	9^h35^m	16^h31^m2 *a*	23^h30^m
Greenwich Hour Angle	100°44′.9	68°05′.2	310°06′.0
Assumed longitude W.	67°00′.1	68°04′.4	68°54′.0
Local hour angle	33°44′.8	*b*	241°12′.0
Observed altitude	45°13′.9	53°37′.9	41°38′.0
Corrections:			
Refraction and reduction	−1′.0	15′.3	−1′.1
Horizon dip	−4′.6	−4′.6	−4′.6
Corrected altitude	45°08′.3	36°11′.4 *c*	41°32′.3
Declination of sun		6°56′.1	
Hour angle correction	−49′.8	0′.0	+29′.3
Latitude	44°18′.5	43°07′.5	42°01′.6

a Local Civil Time of Local Apparent Noon (transit of sun across meridian) is equal to 12^h00^m, *minus* Equation of Time. Equation of Time at noon, September 5, 1943, is + $1^m.1$. Hence, L.C.T. of App. Noon is $11^h58^m.9$. G.C.T. − L.C.T. for longitude 68°04′.4 = $4^h32^m.3$. Hence, G.C.T. of L.A.N. is $16^h31^m.2$.

b The discrepancy of 00′.8 is due to dropped decimals in computing the time.

c Zenith distance.

We take this position as 67°05′.7 W, 44° N, the assumed longitude leading to a local hour angle expressed as a whole number of degrees (See Table E, column 1) so that the solution of the astronomical triangle involved can be located immediately in H.O. 214, where we find, for Lat. 44°, H.A. 77° and declination +17°01′.4, the altitude, $H_c = 20°57'.2$, and azimuth in Table E.

TABLE E

SIGHTS FOR SUMNER LINES

Computations

Object Observed	Jupiter	Sun (L.L.)	Moon (L.L.)	Alpheratz
Watch time	5^h05^m	14^h30^m	14^h35^m	19^h00^m
G.C.T.—W.T.	4^h30^m	4^h30^m	4^h30^m	4^h30^m
Greenwich Civil Time	9^h35^m	19^h00^m	19^h05^m	23^h30^m
Greenwich Hour Angle	350°05′.7	105°17′.7	40°25′.3	335°35′.7
Assumed longitude	67°05′.7	68°17′.7	68°25′.3	68°35′.7
Local hour angle	77° E	37° E	28° E	93° E
Assumed latitude	44° N	43° N	43° N	42° N
Declination of object	+17° 01′.4	+6° 56′.1	−12° 56′.6	+28° 46′.7
Calculated (H.O.214)				
Altitude H_c	20°59′.0	41° 27′.8	28°27′.7	16°44′.3
Azimuth	86°.3 W	127° .1 E	148°.7 W	66°.1 W
Observation of altitude				
Eye height	24 ft.	24 ft.	24 ft.	24 ft.
Temperature (°F.)	61	75	75	72
Observed altitude	21°08′.8	41°24′.8	27°44′.7	16°43′.7
Refraction and reduction	−2′.5	+14′.9	+63′.8	−3′.3
Horizon dip	−4′.6	−4′.6	−4′.6	−4′.6
Corrected altitude H_o	21°01′.7	41°35′.1	28°43′.9	16°35′.8
H_c	20°59′.0	41°27′.8	28°27′.7	16°44′.3
$H_o - H_c$	+2′.7	+7′.3	+16′.2	−8′.5
Parallax			56′.9	
Quadrant observed	E	SW	SE	NE

The observed altitude for Jupiter, 21°08′.8, corrected for refraction and horizon dip, gives us $H_o = 21°01'.7$. The difference, $H_o - H_c = 2'.7$, is laid off as 2.7 miles from the assumed position, 67°05′.7 W, 44° N along the azimuth of Jupiter. At the point so located, the Sumner line is drawn perpendicular to the line of azimuth (See Fig. 23). This line of position intersects the Polaris line (or Lat. 44°18′.5 N) at the ship's position at 5^h05^m.

ii The noon observation of the sun gives us a latitude. The latitude as so determined, 43°07′.5 N (Table D, column 2), agrees closely with the D.R. latitude for noon (Table C), 43°08′.1 N, and no change is required in the chart.

iii The afternoon observations of sun and moon are reduced for Sumner lines (Table E, columns 2 and 3), for the sun at 14^h30^m and the moon at 14^h35^m. Shifting the sun line parallel with itself a distance equal to 1 mile (corresponding with 5 minutes) along the bearing of the course gives us a running fix of the ship's position at 14^h35^m (See Fig. 25). This position differs somewhat from the D.R. position at 14^h35^m (Table C, column 2) and a new course is plotted from the new fix, but on the same bearing, 212°.

iv The evening sights again correct the position (last columns of Tables C, D, and E, and Fig. 25). Alpheratz is visible in the east. A light fog lies to the west. From this fix, 68°49′ W, 42°02′ N, a new course is laid out on the plotting sheet, bearing 212° as before. The lookout is told to keep a sharp watch for Great Point Light (70°03′ W, 41°23′ N) on the starboard bow.

APPLICATIONS TO AVIATION

It is one thing to plot a course for flying in "still air" (which almost never exists) and quite another thing to figure one's course in an airplane when the wind is blowing. Obviously, the force of the wind must be taken into account as well as the speed of the plane. The direction from which the wind is blowing is also of great importance, as the wind may either help or hinder the airplane in its progress toward its destination.

Wind drift

In taking into account the force and direction of the wind, we must find the combined effect of the travel of the airplane through the air and the travel of the air over the ground. Here we have at least three distinct cases: when the wind is traveling in the direction which we wish the plane to pursue; when the wind is traveling in a direction exactly opposite the direction of the plane; and when the wind is blowing across the path of the plane. We shall consider each of these cases separately.

WITH THE WIND

When the plane is traveling in the direction toward which the wind is blowing, simple arithmetic addition gives us the solution to our problem. If we could imagine an object suspended in air making no speed at all, but offering no resistance to the wind, we could readily see that it would be blown in the direction in which the wind was traveling and at the same rate of speed as the wind. Since the airplane does possess speed of its own, we realize that its actual speed, when it is traveling with the wind, represents its own speed plus the speed of the wind.

Illustrative Problem

Given an airplane traveling at a speed of 95 knots, heading with the wind blowing in the same direction at 25 knots, find the distance the pilot may expect his plane to travel in 3 hours.

In Fig. 26, AP = speed of plane; PW = velocity of wind.

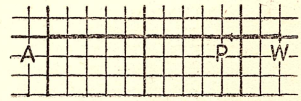


Fig. 26

$$\begin{pmatrix}\text{Speed of plane}\\ \text{plus}\\ \text{speed of wind}\end{pmatrix} \times \begin{matrix}\text{number}\\ \text{of}\\ \text{hours}\end{matrix} = \begin{matrix}\text{distance}\\ \text{traveled.}\end{matrix}$$

$$(95+25) \times 3 = 120 \times 3 = 360 \text{ miles.}$$

AGAINST THE WIND

If the wind is blowing in a direction exactly opposite to that traveled by the plane, we follow the same line of reasoning, but this time subtract the rate of the wind from the rate of the plane. Since the wind is blowing against the airplane, we realize that the actual speed of the plane will be *reduced* by the rate of speed of the wind.

Illustrative Problem

Given an airplane traveling at a speed of 95 knots, heading against a wind blowing 25 knots, find the distance the pilot may expect his plane to travel in 3 hours.

In Fig. 27, AP = speed of plane; PW = velocity of wind.

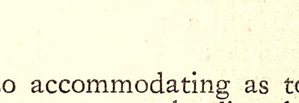


Fig. 27

$$\begin{pmatrix}\text{Speed of plane}\\ \text{minus}\\ \text{speed of wind}\end{pmatrix} \times \begin{matrix}\text{number}\\ \text{of}\\ \text{hours}\end{matrix} = \begin{matrix}\text{distance}\\ \text{traveled.}\end{matrix}$$

$$(95-25) \times 3 = 70 \times 3 = 210 \text{ miles.}$$

WIND VECTORS

We do not always find that the wind is so accommodating as to travel in the direction which we wish to take, nor even to be directly contrary to the course we wish to pursue. When the wind is blowing against the course, we have to resort to more complicated figuring to determine the *track*, or course made good. This we may do by plotting the *wind vector*, making use of what we learned about the parallelogram of forces on page 501.

Illustrative Problem

Given an airplane traveling at a speed of 95 knots due northeast (*i.e.*, with a heading of 45°) with the wind blowing from the direc-

tion, 270°, at 25 miles an hour. In what direction has the plane actually flown and what was its ground speed?

a We plot the vector representing the speed of the plane as follows: Selecting a point, A, we draw a perpendicular line, AN, to represent the direction of north. (Fig. 28.)

b Measuring an angle of 45° to represent the heading of the plane, we draw a line, AP, selecting any convenient scale and measuring off a distance of 95 units, representing the speed of the plane.

c At the point, P, which is at a distance of 95 units from A, we draw PN' parallel to AN.

d Since the wind is blowing from the direction, 270°, we subtract this number from 360°, getting 90° as the angle at which we wish to draw the wind vector.

e Measuring 90° from PN' at P, we draw PW 25 units long to represent the rate the wind is traveling (the wind vector).

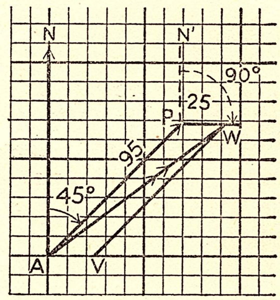


Fig. 28

f By drawing AV parallel to PW and VW parallel to AP, we complete a parallelogram.

g The diagonal of the parallelogram, AW, gives us the track of the plane. Since we have drawn our lines to scale, we may readily measure the length of AW and determine that the plane has actually flown over the ground at a rate of 117 knots. Measuring $\angle NAW$ gives us 55° as the actual course of the plane.

To save ourselves time, we may omit step f and compute directly the value of AW. When we do this, we speak of drawing a *triangle of velocities* instead of a parallelogram of velocities.

DETERMINING THE COURSE TO STEER

We may make use of this same device if we wish to determine what course should be steered to reach a given point at a given time.

Illustrative Problem

Given a track of 210° and a wind coming from the direction, 320°, at 30 miles per hour, determine the true course which an airplane traveling 120 miles per hour should steer, and compute its ground speed.

a Draw AN to represent the line to the north, as before.

b Draw an indefinite line, AT, to represent track, at an angle of 210° (or 30° from NA extended). $NA = 180°$. $210° - 180° = 30°$.

c Draw UV through A at an angle of 320° with AN.

d On AV, measure AW equal to 30 units, representing the velocity of the wind.

e With W as a center, and a radius of 120 units (the speed of the plane), describe an arc, cutting AT at C.

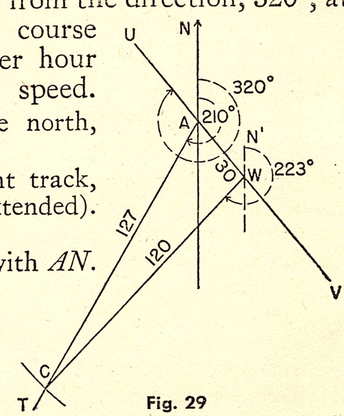


Fig. 29

f Draw WN' parallel to AN.

g Connect W and C. Measurement of AC gives us a speed of 127 miles an hour. Measurement of $\angle N'WC$ gives us 222° as the course to steer.

Radius of action

When a pilot leaves an airplane carrier on a scouting mission, he must determine how far he can safely fly along a given course and in what direction he must fly back on his return trip to the carrier, which is also in motion during his flight. He first determines how much gasoline he has in his tank and how many hours' flight that much "gas" will permit him. In order to have a margin of safety, he holds at least one hour's fuel in reserve, basing his computation on the remainder.

If the carrier were anchored and no wind were blowing, it is obvious that the plane could fly half the distance its fuel supply permitted and return to the same spot within the permitted flying time. When the carrier is in motion and there is a wind, the problem becomes somewhat more complicated.

Illustrative Problem

Given an airplane carrier steaming on a course of 70° at 22 miles an hour, how far can a pilot fly a plane with 3 hours' gas supply in the direction, 190°? (Speed of plane, 120 m.p.h.) Determine the point at which he must turn back and the time at which he will reach that point. (Assume no wind.)

a Since the carrier will travel 22×3 ($=66$ miles), we plot the line, CD, to represent the distance covered by the carrier while the plane is in flight. (Fig. 30.)

b Draw CP with $\angle NCP = 190°$ to represent the proposed course of the plane.

c On CP, indicate point S as the spot to which the plane could fly in 3 hours.

d Draw DS, and find its perpendicular bisector. (See page 282.)

e Note the point, R, where the bisector, AB, of DS meets CS.

f Draw RD, which represents the return journey of the plane from the point, R, to the carrier at its new position at D.

g Since $RD = RS$, it is evident that this line still represents the safe return flying distance for the plane.

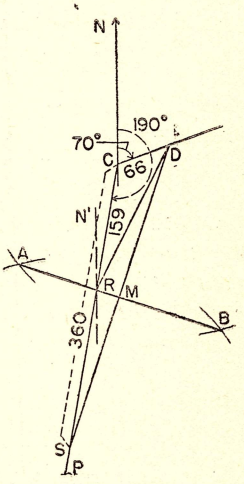


Fig. 30

h Measurement of CR gives us the distance, 159 miles, which the plane may fly on its course.

i Measurement of $\angle N'RD$, 26°, gives us the direction of the return flight.

APPLICATIONS TO RADIO

In figuring the voltage used in the detector circuit of some television receivers, we have to bring some knowledge of trigonometry into play. Given the circuit represented in Fig. 31, we may write these equations:

$$e_3 = e_1 + e_p = E_1 \sin \omega t + E_p \cos \omega t \qquad \textbf{a}$$

$$e_4 = e_2 + e_p = E_2 \sin \omega t + E_p \cos \omega t \qquad \textbf{b}$$

$$e_1 = -e_2 \qquad \textbf{c}$$

Substituting c in b, we have

$$\therefore \quad e_4 = -e_1 + e_p = -E_1 \sin \omega t + E_p \cos \omega t. \qquad \textbf{d}$$

The facts that e_1 and e_2 are 180° out of phase and that e_1 and e_p are 90° out of phase are justified by experimental work and by the analysis of the action of the transformer.

It would be desirable if we could express e_3 and e_4 as single sinusoidal waves rather than by the summations given above. By letting

$$E_1 = E_3 \sin \theta, \qquad \textbf{e}$$

and

$$E_p = E_3 \cos \theta, \qquad \textbf{f}$$

we get

$$E_3 = \frac{E_1}{\sin \theta} \qquad \textbf{e}'$$

$$\sin \theta = \frac{E_1}{E_3} \qquad \textbf{e}''$$

$$\sin^2 \theta = \frac{E_1^2}{E_3^2} \qquad \textbf{e}'''$$

$$E_3 = \frac{E_p}{\cos \theta} \qquad \textbf{f}'$$

$$\cos \theta = \frac{E_p}{E_3} \qquad \textbf{f}''$$

$$1 - \sin^2 \theta = \frac{E_p^2}{E_3^2} \qquad \textbf{f}'''$$

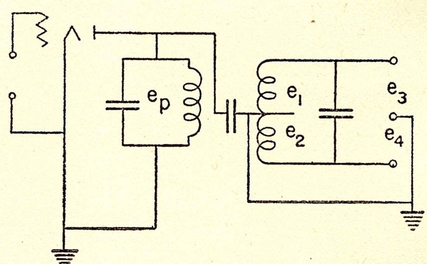


Fig. 31

Adding e‴ and f‴, we get

$$1 = \frac{E_1^2 + E_p^2}{E_3^2} \qquad \textbf{g}$$

$$E_3^2 = E_1^2 + E_p^2 \qquad \textbf{g}'$$

We could just as readily have obtained this equation by squaring and adding e and f:

$$E_1^2 = E_3^2 \sin^2 \theta \qquad \textbf{e}^2$$

$$E_p^2 = E_3^2 \cos^2 \theta \qquad \textbf{f}^2$$

$$E_1^2 + E_p^2 = E_3^2 (\sin^2 \theta + \cos^2 \theta). \qquad \textbf{e}^2 + \textbf{f}^2$$

Since

$$\sin^2\theta + \cos^2\theta = 1,$$

$$E_1^{\,2} + E_p^{\,2} = E_3^{\,2} \qquad \mathbf{g'}$$

In future operations, then, the reader may proceed directly to this point.

From this, $$E_3 = \sqrt{E_1^{\,2} + E_p^{\,2}}. \qquad \mathbf{g''}$$

Now $$\tan\theta = \frac{E_1}{E_p} \qquad \mathbf{h}$$

or $$\theta = \tan^{-1}\frac{E_1}{E_p} \qquad \mathbf{h'}$$

(Any finite values—positive or negative—assigned to E_1 and E_p result in finite values for E_3 and θ; hence, these substitutions are permissible.)

By trigonometric identities, we find that

$$e_3 = E_3 \cos(\omega t - \theta) \qquad \mathbf{i}$$

$$e_4 = E_3 \cos(\omega t + \theta) = E_3 \cos(\omega t - \gamma) \qquad \mathbf{j}$$

where $$\gamma = \tan^{-1}\frac{-E_1}{E_p}.$$

Since the *average* values of e_3 and e_4 are identical in value,

$$e_3 + e_4 = 2\,e_3 \qquad \mathbf{k}$$

or $$e_3 + e_4 = 0, \qquad \mathbf{k'}$$

depending upon the direction of their polarities when added.

With a change in the frequency of the incoming signal, the secondary voltage shifts in phase and results in these equations:

$$e_3 = E_1 \sin(\omega t + \beta) + E_p \cos\omega t \qquad \mathbf{l}$$

$$e_4 = -E_1 \sin(\omega t + \beta) + E_p \cos\omega t. \qquad \mathbf{m}$$

By trigonometry,

$$e_3 = E_1 \sin\omega t \cos\beta + E_1 \cos\omega t \sin\beta + E_p \cos\omega t$$

$$= (E_1 \cos\beta)\sin\omega t + (E_1 \sin\beta + E_p)\cos\omega t \qquad \mathbf{n}$$

where

$$E_3 = \sqrt{(E_1\cos\beta)^2 + (E_1\sin\beta + E_p)^2}$$

$$= \sqrt{E_1^{\,2} + E_p^{\,2} + 2E_1E_p\sin\beta}$$

and $$\theta = \tan^{-1}\frac{-E\cos\beta}{-E\sin\beta\,E_p}.$$

Similarly,

$$e_4 = -E_1 \sin\omega t \cos\beta - E_1 \cos\omega t \sin\beta + E_p\,\omega t$$

$$= E_4 \cos(\omega t - \gamma),$$

where, in this case,

$$e_4 = \sqrt{(-E_1\cos\beta)^2 + (-E_1\sin\beta + E_p)^2}$$

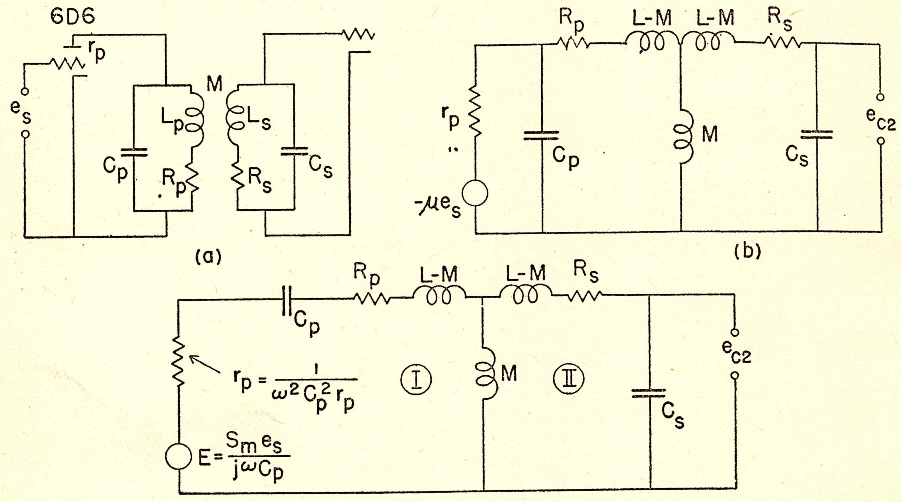
$$= \sqrt{E_1^{\,2} + E_p^{\,2} - 2\,E_1E_p\sin\beta}.$$

and $$\gamma=\tan^{-1}\frac{-E_1\cos\beta}{-E_1\sin\beta+E_p}.$$

Since E_3 and E_4 are now different in magnitude, it is evident that the average values of e_3 and e_4 will now be different. Thus, a change in the phase relationships of the incoming signals results in a change in the amplitude of the sum (e_3+e_4). This changed amplitude may now be used to control the bias potential on amplifier stages (A.F.C.) or it may in itself be the signal which we want (F.M. signals).

Illustrative Problem

To demonstrate the use of mathematics in a particular case, the circuit of Fig. 32 has been selected. This might be the intermediate amplifier stage of a superheterodyne receiver. The equivalent circuits drawn are required for the analysis,



(a) (b) (c)

Fig. 32

where: **a** the inductance of the primary, L_p, equals the inductance of the secondary, L_s, equals 4 millihenries;
b the frequency to which the circuits are tuned is 260 kc.;
c the figure of merit of the primary, Q_p, equals the figure of merit of the secondary, Q_s, equals 50;
d the coefficient of coupling between L_p and L_s is 0.03.

From the resonant frequency formula, $\omega=\frac{1}{\sqrt{LC}}$, we calculate the values of C_s and C_p:

$$C_s=C_p=\frac{1}{4\pi^2f^2L}=\frac{1}{4\pi^2\times260^2+10^6\times4\times10^{-3}}=93.4\ \mu\mu f.$$

From the formula for "figure of merit", $Q=\frac{\omega L}{R}$, we calculate the values of R_p and R_s:

$$R_p=R_s=\frac{\omega L}{Q}=\frac{2\pi\times260\times10^3\times4\times10^{-3}}{50}=131\text{ ohms}.$$

From the formula, $M = k\sqrt{L_pL_s}$, we may calculate the mutual inductance in the circuit:

$$M = k\sqrt{L_pL_s} = 0.03 \times 4 \times 10^{-3} = 0.12 \times 10^{-3} \text{ henries.}$$
$$L - M = 4 \times 10^{-3} - 0.12 \times 10^{-3} = 3.88 \times 10^{-3} \text{ henries.}$$

From the formulas for the equivalent circuit elements in Fig. 16c:

$$E = e_s \frac{1600 \times 10^{-6}}{j2\pi \times 260 \times 10^3 \times 93.4 \times 10^{-12}} = 10.5\, e_s$$

$$r'_p = \frac{1}{4\pi^2 \times 260^2 \times 10^6 \times (93.4)^2 \times 10^{-24} \times 0.8 \times 10^6} = 53.5 \text{ ohms.}$$

$$I_3 = \frac{E\omega M}{Z_{11}Z_{22} - Z_{12}^2}.$$

At 260 kc, the primary and secondary circuits are "tuned",

$$Z_{11} = r'_p + R_p = 53.5 + 131.5 = 184.5 + j0 \text{ ohms}$$
$$Z_{12} = 0 + j\omega M = 0 + j196 \text{ ohms}$$
$$Z_{22} = R_s + j0 = 131 + j0 \text{ ohms.}$$

The gain of the coupling circuit is

$$\frac{\omega M X_{Cs}}{Z_{11}Z_{22} - Z_{12}^2} = \frac{196 \angle 90° \times 6550 \angle -90°}{184.5 \angle 0° \times 131 \angle 0° - 196^2 \angle 180°} = \frac{1.285 \times 10^6 \angle 0°}{2.42 \times 10^4 + 3.84 \times 10^4} = 20.5$$

Solutions to Questions and Exercises in Issue 13

NAVIGATION

SAILINGS

1 11^h30^m
2 14° W of S
3 (a) 54.06
(b) 98
(c) 112.15
4 Lat. 48°37′ N
Long. 69°15′.5 W
5 350°2′.5
6 329°30′

PILOTING

7 117.3 naut. mi.
8 29°

SPHERICAL TRIGONOMETRY

9 $\cos c = \cot A \cot B$
10 2,480 naut. mi.
2,852 stat. mi.
11 58′ too large
12 81°19′
13 2,627 naut. mi.

GREAT-CIRCLE SAILING

14 250
15 0.2 mi.
16 7°57′

CELESTIAL COÖRDINATES

17 9^h7^m
18 48°18′
19 $7^h41^m50^s$
20 $5^h55^m8^s$
21 52°57′ W
22 $1^h27^m16^s$
23 $2^h02^m14^s$

LINES OF POSITION

24 140° W 0° N, 140° W 30° N, 155°30′ W 15° N, 124°30′ W 15° N

WIND TRIANGLE

25 77°22′
26 46 mph
27 35 mph

RADIO

1 $r \sin\theta + s \cos\theta = t$
$m \sin\theta \sin\gamma + m \cos\theta \cos\gamma = t$
(Using trig formula for cos of difference of 2 angles)
$m \cos(\theta - \gamma) = t$

2 455 meters

3 $n = \dfrac{19.96M\sqrt{l^2 + A^2}}{Na^2}$

4 $e = E \sin \omega t$; $i = I \sin \omega t$
$p = e\,i = EI \sin^2 \omega t = EI(\frac{1}{2} - \frac{1}{2} \cos 2\omega t)$
Average power $P = \int_0^1 p\,dt = \frac{1}{2}EI$

5 Equivalent impedance = sum of individual impedances =
$(8.7 + 5j) + (10.6 + 10.6j) + (10 + 17.3j) = (29.3 + 28.4j)$

6 (*Note*: $l = np$)
$r^2n^2 - 9rL - 10pLn = 0$

$$n = \frac{10pL \pm \sqrt{(10pL)^2 + 36r^3L}}{2r^2}$$
$$= \frac{10 \times \frac{3}{8} \times 10 \pm \sqrt{(\frac{75}{2})^2 + 36(\frac{51}{32})^3 10}}{2(\frac{51}{32})^2}$$
$= 17.93$ or 18 turns

7 Equal
8 $R = 2.2$ ohms
9 $I = 0.57$ ampere
10 Error = 0.06%

11 $R = \dfrac{108 \times 10^6}{0.087} \times \dfrac{10^6}{2.3 \log \frac{127}{113}}$
$= 10.6 \times 10^{15}$ ohms

Index to Practical Mathematics

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Errata and Addenda

PRACTICAL MATHEMATICS

Answers for Issue Number One
Printed in Issue Number Two

BASIC ARITHMETIC

Problem	Correct Answer
35	37,421
56	2,203
61	5,241
87	16,800
119	$160\frac{1}{13}$

FRACTIONS

Problem	Correct Answer
276	$1'11\frac{29}{32}''$
277	$\frac{13}{32}''$
282	$2\frac{3}{4}''$
283	27
284	927
291	$59{,}572\frac{73}{128}$
296	$68\frac{7}{40}$

NUMBERS THROUGH THE AGES

Problem	Correct Answer
9	1896

MEASURING ROD

Problem	Correct Answer
7	3,547
16	40,991
26	$1{,}059\frac{13}{14}$
28	$76\frac{268}{299}$
31	$2\frac{2}{3}$
33	$287\frac{5}{24}$
35	$137\frac{1}{48}$
37	$210\frac{7}{12}$

OMITTED ANSWERS—NUMBERS THROUGH THE AGES

21	0.5	26	0.08
22	0.07	27	0.0045
23	0.000044	28	10,002.5
24	0.01325	29	3.00
25	32.58	30	0.0725

Answers for Issue Number Two
Printed in Issue Number Three

FURTHER PRINCIPLES OF ARITHMETIC

Problem	Correct Answer
5	6.708
12	0.541
17	0.8
32	0.94
40	154.28
55	0.0625
114	15.2
129	428 yd.
147	8587.5 sq. in.

LOGARITHMS

Problem	Correct Answer
7	1
44	25.99
46	541.2
49	1300
55	32.10
64	9.02351
71	3345×10^{10}

SLIDE RULE

Problem	Correct Answer
16	376,300
26	541,000
27	1.602
28	3,130
29	2,620,000
36	45
37	245
39	9.76

MEASURING ROD

Problem	Correct Answer
1	4.57″
23	9.456
35	36.6 ft.
40	379.8
45	774,400 sq. yd.
50	5 hr. 55 min. per day
51	1,490 sq. ft.
56	$22\frac{6}{7}'$

OMITTED ANSWERS—FURTHER PRINCIPLES

148	$6\frac{1}{3}'' \times 8\frac{2}{3}''$
149	11.97 tons
150	$\frac{3}{4}$
151	1:2.56
152	72

OMITTED ANSWER—SLIDE RULE

41	6.77

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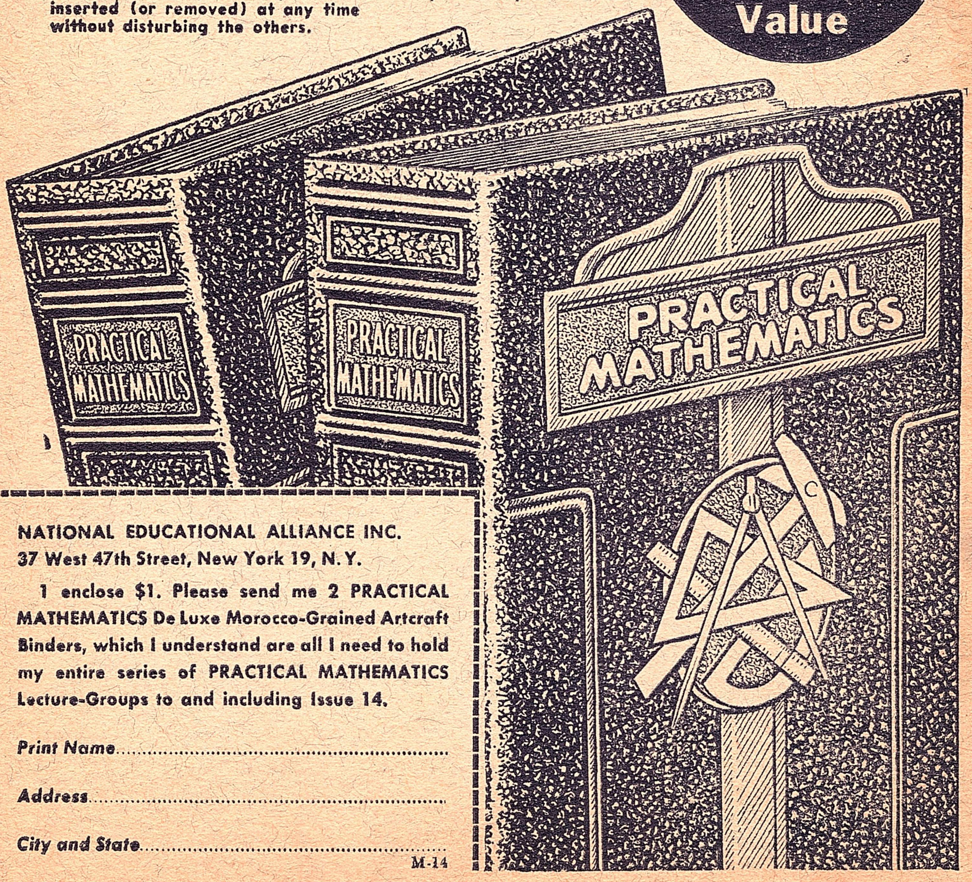

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